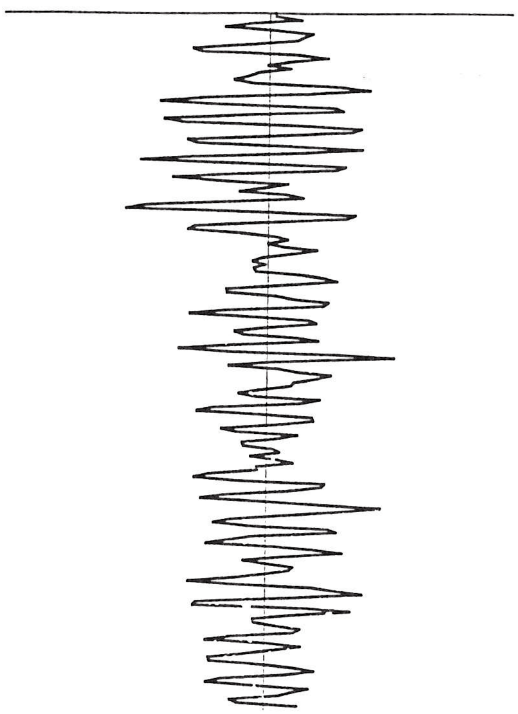
THE RANDOM PROCESSES TUTOR:

A COMPREHENSIVE SOLUTIONS MANUAL FOR INDEPENDENT STUDY

to accompany *Introduction to Random Processes with Applications to Signals and Systems*

William A. Gardner and Chih-Kang Chen



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PREFACE

This solutions manual was written to accompany the book *Introduction to Random Processes with Applications to Signals and Systems* as a study aid. Graduate-level courses in random processes, for which this book was written, require from the student a considerable degree of mathematical sophistication. Perhaps more than any other subject in engineering and science, random processes draws from a wide variety of areas in mathematics and is by its nature a highly analytical subject. For most students, a course in random processes is an unusually challenging experience that results in a substantial increase in mathematical maturity, but not without a considerable investment of effort. This effort primarily takes the form of solving problems and doing exercises that involve the concepts and analytical tools of the subject. Although a certain degree of frustration with challenging problems is a necessary part of the learning process, this frustration must be resolved. Typically, this requires some intervention by instructors or tutors. And this is the intended role of *The Random Processes Tutor*. With the aid of this detailed solutions manual, the student can successfully and enjoyably master the subject of random processes.

Because the subject of random processes draws from so many areas within mathematics, such as algebra, geometry, trigonometry, calculus, combinatorics, logic, probability, and real analysis, as well as from various more specialized topics, such as Fourier analysis and linear system theory, most textbooks on this subject necessarily assume that the reader has broad mathematical skills. But in many cases, this assumption is optimistic. For this reason, *The Random Processes Tutor* has been written to assist the student in refreshing skills that have been learned in the past but not recently exercised.

Although there are often a variety of approaches to solving a specific problem, elegant solutions are preferable because they demonstrate depth of understanding, not to mention their aesthetic appeal. When various approaches to a solution were evident to us, we chose the one we considered to be more elegant. Every solution presented in this manual has been thoroughly checked by both authors. Nevertheless, in over 250 pages of solutions, some errors are inevitable. For this, we apologize.

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William Gardner expresses his gratitude to Mr. Richard Smith and Coauthor Dr. Chih-Kang Chen for their preparation of errata and insertion into this Re-Release.

A NOTE TO INSTRUCTORS

For some instructors, the first impression of having a complete solutions manual available to students will be a negative one. But upon closer examination of the substantial advantages, we believe that impression will change.

To begin with, development of the skill of independent study is one of the major goals of every graduate program. As explained in the Preface, this manual makes independent study of the highly analytical subject of random processes practically feasible for all levels of graduate students -- not just the top 5 or 10 percent. Independent study here means study based only on the textbook, this manual, possibly some remedial references, and preferably (but not necessarily -- at least for some students) lectures. Thus, problem sessions, extensive office hours, tutors, homework evaluations, and so on all can be done away with, and the student will naturally learn to become more independent without suffering from neglect. Moreover, the lectures can have more flexibility because of time not required for explaining how to approach and solve the problems in the textbook; although this *can* be done (with ease) with the aid of this manual if desired.

Of course, this also solves the perennial problem of finding enough instructor time and/or sufficiently qualified assistants to perform the tasks associated with problem sessions, office hours, tutoring, and homework evaluation.

Another advantage is that much less reliance on an honor code is necessary, because students are not evaluated on the basis of their success in doing assigned homework problems. Evaluation can be based entirely on examinations and, if desired, short quizzes, term projects, and so on.

With this manual, the instructor's only task regarding homework is to select which of the over 350 exercises in the textbook the students should focus on. Instructors who use frequent due-dates for homework to encourage students not to fall behind can, instead, use periodic brief quizzes on material from homework (since, of course, there is no homework due when this manual is used).

HOW TO USE THIS MANUAL

As we all know, problem solving and struggling to grasp new concepts typically involve frustration; as we also know, it is natural to try to avoid or minimize frustration. Thus, there is the danger that with a comprehensive solutions manual like this one at hand, the student will eagerly look up the solution to assigned exercises as soon as each problem statement has been read. Although some learning would indeed result from this approach, it would not be deep and lasting. Rather it would tend to be shallow and soon forgotten (maybe even before the final examination for the course!) and, sadly, the excitement of challenge and the joy of discovery would be lost.

The depth and the longevity of learning seem to be closely linked to the size of the reward felt upon discovering the solution to a problem tackled. The size of the reward felt depends on the magnitude of the challenge, which affects the amount of time and effort invested in the process of seeking the solution. Simply put, large profits require sizable investments.

It may seem to some students that understanding the problem statements in some of the exercises in the textbook, and then understanding the solutions given in this manual requires enough investment of effort and does indeed feel rewarding. But this involves almost no challenge: the rewards gained in discovering the solutions by oneself are so much greater. This is so even if one's own solution is only partially complete, or partially correct, or can otherwise be refined or improved on once the solutions given in this manual have been read. And there is always the possibility of finding a better solution than that given in this manual.

Therefore, please take our advice: In trying to solve the assigned problems that you find particularly challenging, invest all the time and effort you can. Do not look up a solution in this manual until you have either succeeded in finding a solution yourself, or your progress through the textbook would be seriously hindered by not understanding how to solve a particular problem.

Chapter 1

Probability and Random Variables

1.1 The given set combinations can be reduced as follows:

a) $(A\cap\bar{B})\cap(B\cap\bar{A}) = A\cap(\bar{B}\cap B)\cap\bar{A} = A\cap\varnothing\cap\bar{A} = \varnothing,$

b) $(A\cap\bar{B})\cup(A\cap B) = A\cap(\bar{B}\cup B) = A\cap S = A,$

c) $\overline{A\cap(B\cup C)} = \overline{(A\cap B)\cup(A\cap C)} = \overline{(A\cap B)}\cap\overline{(A\cap C)} = (\bar{A}\cup\bar{B})\cap(\bar{A}\cup\bar{C}),$

d) $\overline{A\cap B\cap C} = \overline{A\cap B}\cup\bar{C} = \bar{A}\cup\bar{B}\cup\bar{C}.$

1.2 Simply draw a Venn diagram of three intersecting sets A, B, and C, and then add and subtract areas.

1.3 The desired probabilities can be obtained as follows:

a) $$Prob = \frac{m}{n} = \frac{5}{100} = 0.05,$$

b) $$Prob = \frac{m-1}{n-1} = \frac{4}{99} = 0.0404,$$

c) (i) $$Prob = \frac{m}{n}\times\frac{m-1}{n-1} = \frac{5}{100}\times\frac{4}{99} = 0.00202,$$

(ii) $$Prob = \frac{m}{n}\times\frac{n-1-(m-1)}{n-1} + \frac{n-m}{n}\times\frac{m}{n-1}$$
$$= \frac{5}{100}\times\frac{99-4}{99} + \frac{100-5}{100}\times\frac{5}{99} = 0.096,$$

(iii) $$Prob = \frac{n-m}{n}\times\frac{(n-1)-m}{n-1} = \frac{100-5}{100}\times\frac{99-5}{99} = 0.902.$$

1.4 *a*) Let D denote the event that one defective circuit is selected, and let A, B, and C denote the events that types A, B, and C circuits, respectively, are selected from the mix. Then

$$\begin{aligned}P(D) &= P(D\cap[A\cup B\cup C])\\ &= P(D\cap A) + P(D\cap B) + P(D\cap C)\\ &= P(D\,|\,A)P(A) + P(D\,|\,B)P(B) + P(D\,|\,C)P(C)\\ &= \frac{1}{10}\times\frac{100}{600} + \frac{1}{20}\times\frac{200}{600} + \frac{1}{30}\times\frac{300}{600} = \frac{1}{20}.\end{aligned}$$

b) Given the event D, we obtain the probability of event A as follows:

$$P(A \mid D) = \frac{P(A \cap D)}{P(D)}$$

$$= \frac{P(D \mid A)P(A)}{P(D \mid A)P(A) + P(D \mid B)P(B) + P(D \mid C)P(C)} = \frac{1}{3}.$$

1.5 Let the event that the first series resistor shorts be denoted by A, and the let the event that the single resistor in one parallel branch shorts be denoted by B, and let the events that each of the resistors in the other parallel branch shorts be denoted by C and D. Then

$$Prob\{\text{short}\} = P(A \cap [B \cup [C \cap D]]) = P(A)[P(B) + P(C \cap D) - P(B \cap [C \cap D])]$$

$$= p(p + p^2 - p^3).$$

1.6 For any set of n events $\{A_1, A_2, \ldots, A_n\}$, we have

$$P(A_1 \cup A_2 \cup \cdots \cup A_n) = P(A_1) + P(A_2 \cup \cdots \cup A_n) - P(A_1 \cap [A_2 \cup \cdots \cup A_n])$$

$$\leq P(A_1) + P(A_2 \cup \cdots \cup A_n)$$

$$\vdots$$

$$\leq P(A_1) + P(A_2) + \cdots + P(A_n).$$

1.7 *a*) The probability of instability is given by

$$P(A \cap B) = P(B \mid A)P(A) = \frac{1}{30000}.$$

b) The conditional probability of instability is given by

$$P(A \mid B) = P(A \cap B)/P(B) = \frac{1}{150}.$$

1.8 *a*) Since $\int_{-\infty}^{\infty} f_Y(y)dy = 1$, then for the given probability distribution it can be shown that $a = b$.

b) Since the desired conditional probability density is given by

$$F_{Y|Y>c>0}(y) = P(Y < y \mid Y > c)$$

$$= \frac{P(Y < y \text{ and } Y > c)}{P(Y > c)} = \frac{P(c < Y < y)}{P(Y > c)}$$

$$= \frac{\int_c^y f_Y(u)dy}{\int_c^\infty f_Y(u)du}, \qquad y > c,$$

then the conditional probability density is given by

$$f_{Y|Y>c>0}(y) = \frac{ae^{-ay}}{\int_c^\infty ae^{-ax}\,dx} = ae^{-a(y-c)}, \qquad y > c.$$

c) Similarly, the desired conditional distribution is given by

$$F_{Y|0<c<Y<d}(y) = P(Y < y)|0 < c < Y < d)$$

$$= \frac{P(c < Y < y < d)}{P(c < Y < d)} = \frac{\int_c^y f_Y(u)du}{\int_c^d f_Y(u)du}$$

$$= \frac{e^{-ac} - e^{-ay}}{e^{-ac} - e^{-ad}}, \qquad c < y < d$$

and the corresponding density is given by

$$f_{Y|0<c<Y<d}(y) = \frac{ae^{-ay}}{\int_c^d ae^{-ax}\,dx} = \frac{ae^{-ay}}{e^{-ac} - e^{-ad}}, \qquad c < y < d.$$

1.9 *a*) The probability that the digit 1 is received is given by

$$P[Y=1] = P[Y=1|X=0]P[X=0] + P[Y=1|X=1]P[X=1]$$

$$= q_o q + p_1 p.$$

b) The conditional probability of error is given by

$$P[Y=0|X=1] = q_1.$$

c) The unconditional probability of error is obtained as follows:

$$P_{error} = P[Y=0|X=1]P[X=1] + P[Y=1|X=0]P[X=0]$$
$$= q_1 p + q_0 q.$$

1.10 Referring to Figure 1.9, we have the probability density

$$f_X(x) = P_1 \delta(x - V_1) + P_2 \delta(x - V_2).$$

Letting Y be the voltage across the resistor, we have the density

$$f_Y(y) = \frac{1}{\alpha\sqrt{2\pi}} \exp\{-\frac{y^2}{2\alpha^2}\}.$$

Letting Z be the total voltage across the whole circuit, we then have the density

$$f_Z(z) = \int_{-\infty}^{\infty} f_Y(z-s) f_X(s) ds = P_1 f_Y(z - V_1) + P_2 f_Y(z - V_2).$$

1.11 For $Y = (X-b)/a$ and $a > 0$, we have the probability distribution

$$F_Y(y) = Prob\{Y < y\} = Prob\{\frac{X-b}{a} < y\} = Prob\{X < ay + b\} = F_X(ay+b)$$

$$f_Y(y) = \frac{dF_Y(y)}{dy} = \frac{dF_X(ay+b)}{dy} = af_X(ay+b).$$

For $a < 0$, we have the distribution

$$F_Y(y) = Prob\{Y < y\} = Prob\{\frac{X-b}{a} < y\} = Prob\{X > ay + b\} = 1 - F_X(ay+b)$$

$$f_Y(y) = \frac{dF_Y(y)}{dy} = -af_X(ay+b).$$

Thus, the probability density is given by

$$f_Y(f) = |a| f_X(ay+b).$$

1.12 *a*) (i) $\alpha = 1$, $\beta = 0$. To show that the area of the probability density is unity, we proceed as follows:

$$I^2 = \iint_{-\infty}^{\infty} \frac{1}{2\pi} \exp\{-\frac{1}{2}(x^2+y^2)\} dxdy = \int_0^{2\pi}\int_0^{\infty} \frac{1}{2\pi} \exp\{-\frac{1}{2} r^2\} r dr d\theta$$
$$= \int_0^{2\pi} \frac{1}{2\pi} d\theta \int_0^{\infty} \exp\{-t\} dt = 1.$$

Thus,

$$I = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\{-\frac{1}{2}x^2\} dx = 1.$$

(ii) $\alpha \neq 1$, $\beta \neq 0$. For this more general case, a change of variables in the area

$$J \triangleq \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\alpha^2}} \exp\{-\frac{1}{2}(\frac{x-\beta}{\alpha})^2\} dx$$

of the density yields

$$J = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\{-\frac{1}{2}z^2\} dz.$$

Then it follows from (i) that $J = 1$.

b) From the hint, we have $dw = \frac{1}{\alpha'} \frac{1}{\sqrt{1-\gamma^2}} dy$. Therefore, using the hint, we have the univariate density

$$\begin{aligned}
f_X(x) &= \int_{-\infty}^{\infty} f_{XY}(x, y) dy \\
&= \int_{-\infty}^{\infty} \frac{1}{2\pi\alpha\alpha'\sqrt{1-\gamma^2}} \exp\{-\frac{1}{2}(\frac{x-\beta}{\alpha})^2\} \exp\{-\frac{1}{2}w^2\} \alpha'\sqrt{1-\gamma^2} dw \\
&= \frac{1}{\sqrt{2\pi}\alpha} \exp\{-\frac{1}{2}(\frac{x-\beta}{\alpha})^2\} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\{-\frac{1}{2}w^2\} dw \\
&= \frac{1}{\sqrt{2\pi}\alpha} \exp\{-\frac{1}{2}(\frac{x-\beta}{\alpha})^2\},
\end{aligned}$$

which is (1.23), as desired.

c) To simplify the proof, let $Z = U + V$, where $U = aX$ and $V = bY$. Then,

$E\{U\} = \beta_U = a\beta, \quad Var\{U\} = \alpha_U^2 = (a\alpha)^2,$

$E\{V\} = \beta_V = b\beta', \quad Var\{V\} = \alpha_V^2 = (b\alpha')^2, \quad Cov\{U, V\} = abCov\{X, Y\},$

from which it follows that $\gamma_{UV} = \gamma$ and

$$f_{UV}(u, v) = \frac{1}{|a||b|} f_{XY}(x, y).$$

It follows from (1.45) that $f_Z(z) = \int_{-\infty}^{\infty} f_{UV}(z-v, v) dv$. Therefore, the probability density is given by

$$f_Z(z) = \frac{1}{2\pi\alpha_U\alpha_V\sqrt{1-\gamma_{UV}^2}}$$

$$\times \int_{-\infty}^{\infty} \exp\left\{-\frac{\left(\frac{z-v-\beta_U}{\alpha_U}\right)^2 - 2\gamma_{UV}\left(\frac{z-v-\beta_U}{\alpha_U}\right)\left(\frac{v-\beta_V}{\alpha_V}\right)+\left(\frac{v-\beta_V}{\alpha_V}\right)^2}{2(1-\gamma_{UV}^2)}\right\} dv.$$

Carrying out the above integration, we obtain

$$f_Z(z) = \frac{1}{\sqrt{2\pi}\sqrt{\alpha_U^2+2\gamma_{UV}\alpha_U\alpha_V+\alpha_V^2}}\exp\left\{-\frac{1}{2}\frac{(z-\beta_U-\beta_V)^2}{\alpha_U^2+2\gamma_{UV}\alpha_U\alpha_V+\alpha_V^2}\right\}$$

$$= \frac{1}{\sqrt{2\pi}\sigma}\exp\{-\frac{1}{2}(\frac{z-m}{\sigma})^2\},$$

which is a Gaussian probability density function (with $m = a\beta - b\beta'$ and $\sigma^2 = (a\,\alpha)^2 + 2\gamma ab\,\alpha\alpha' + (b\,\alpha')^2$).

d) From the definition of conditional probability density, we have

$$f_{X|Y}(x\,|Y=y) = \frac{f_{XY}(x,y)}{f_Y(y)}$$

$$= \frac{1}{2\pi\alpha\alpha'\sqrt{1-\gamma^2}}\exp\left\{-\frac{\left(\frac{x-\beta}{\alpha}\right)^2-2\gamma\left(\frac{x-\beta}{\alpha}\right)\left(\frac{y-\beta'}{\alpha'}\right)+\left(\frac{y-\beta'}{\alpha'}\right)^2}{2(1-\gamma^2)}\right\}$$

$$\times \sqrt{2\pi}\alpha'\exp\{\frac{1}{2}(\frac{y-\beta'}{\alpha'})^2\}$$

$$= \frac{1}{\sqrt{2\pi}\alpha\sqrt{1-\gamma^2}}\exp\left\{-\frac{1}{2}\left(\frac{x-[\beta+\gamma(\frac{y-\beta'}{\alpha'})\alpha]}{\alpha\sqrt{1-\gamma^2}}\right)^2\right\}$$

$$= \frac{1}{\sqrt{2\pi}\alpha''}\exp\{-\frac{1}{2}(\frac{x-\beta''}{\alpha''})^2\},$$

which is a Gaussian density.

1.13 In this case, the probability density of $Z = X + Y$ is given by

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(z-u)f_Y(u)du = \begin{cases} za^{-2}, & 0 \le z < a \\ 2a^{-1} - za^{-2}, & a \le z \le 2a \\ 0, & otherwise. \end{cases}$$

1.14 Let

$$Y = Y_1 = X_1 + X_2 \quad \text{and} \quad Y_2 = X_2$$

or, equivalently,

$$\mathbf{Y} = \mathbf{AX},$$

where

$$\mathbf{Y} = [\,Y_1 \quad Y_2\,]^T, \qquad \mathbf{X} = [\,X_1 \quad X_2\,]^T, \qquad \mathbf{A} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

Then, from (1.41), we have

$$f_{\mathbf{Y}}(\mathbf{y}) = \frac{f_{\mathbf{X}}(\mathbf{A}^{-1}\mathbf{y})}{|\mathbf{A}|} = f_{\mathbf{X}}(\begin{bmatrix} y_1 - y_2 \\ y_2 \end{bmatrix}) = f_{X_1X_2}(y_1 - y_2, y_2) = f_{X_1X_2}(y - x_2, x_2).$$

Thus, from (1.43), we have

$$f_Y(y) = \int_{-\infty}^{\infty} f_{\mathbf{Y}}(\mathbf{y})dy_2 = \int_{-\infty}^{\infty} f_{X_1X_2}(y - x_2, x_2)dx_2$$

for the density of the sum $Y = X_1 + X_2$.

1.15 Define two vectors **Y** and **X** by

$$\mathbf{Y} = [Y_1 \quad Y_2]^T \qquad \text{and} \qquad \mathbf{X} = [X_1 \quad X_2]^T.$$

Then

$$\mathbf{Y} = \mathbf{g}(\mathbf{X}) = \begin{bmatrix} \sqrt{X_1^2 + X_2^2} \\ \tan^{-1}(X_2 / X_1) \end{bmatrix}$$

or, equivalently,

$$\mathbf{X} = \mathbf{g}^{-1}(\mathbf{Y}) = \begin{bmatrix} Y_1 \cos Y_2 \\ Y_1 \sin Y_2 \end{bmatrix}$$

From (1.39), we have

$$f_{\mathbf{Y}}(\mathbf{y}) = f_{\mathbf{X}}\left[\mathbf{g}^{-1}(\mathbf{y})\right]\left|\frac{\partial \mathbf{g}^{-1}(\mathbf{y})}{\partial \mathbf{y}}\right| = f_{X_1}(x_1)f_{X_2}(x_2)\begin{vmatrix} \cos y_2 & -y_1 \sin y_2 \\ \sin y_2 & y_1 \cos y_2 \end{vmatrix}$$

$$= \frac{y_1}{2\pi}\exp\{-\frac{1}{2}y_1^2\}, \qquad y_1 \geq 0, \quad 0 \leq y_2 < 2\pi.$$

The marginal densities are, therefore, given by

$$f_{Y_1}(y_1) = \int_{-\infty}^{\infty} f_{\mathbf{Y}}(\mathbf{y})dy_2 = \int_0^{2\pi} \frac{y_1}{2\pi}\exp\{-\frac{1}{2}y_1^2\}dy_2 = y_1\exp\{-\frac{1}{2}y_1^2\}, \quad y_1 \ge 0$$

$$f_{Y_2}(y_2) = \int_{-\infty}^{\infty} f_{\mathbf{Y}}(\mathbf{y})dy_1 = \int_0^{\infty} \frac{y_1}{2\pi}\exp\{-\frac{1}{2}y_1^2\}dy_1 = \frac{1}{2\pi}, \quad 0 \le y_2 < 2\pi.$$

It follows directly from the preceding that

$$f_{Y_1Y_2}(y_1, y_2) = f_{Y_1}(y_1)f_{Y_2}(y_2),$$

and, therefore, Y_1 and Y_2 are independent.

1.16 Let $\mathbf{X} = [X_1 \quad X_2]^T$ and $\mathbf{Y} = [Y_1 \quad Y_2]^T$. Then, from (1.39), we have

$$f_{\mathbf{X}}(\mathbf{x}) = f_{\mathbf{Y}}(\mathbf{y})\left|\frac{\partial \mathbf{y}}{\partial \mathbf{x}}\right|,$$

where

$$\left|\frac{\partial \mathbf{y}}{\partial \mathbf{x}}\right| = \begin{vmatrix} \dfrac{x_1}{\sqrt{x_1^2+x_2^2}} & \dfrac{x_2}{\sqrt{x_1^2+x_2^2}} \\ \dfrac{-x_2}{x_1^2+x_2^2} & \dfrac{x_1}{x_1^2+x_2^2} \end{vmatrix} = \frac{1}{\sqrt{x_1^2+x_2^2}}.$$

Therefore,

$$f_{\mathbf{X}}(\mathbf{x}) = \frac{y_1}{2\pi}\exp\{-\frac{1}{2}y_1^2\}\frac{1}{\sqrt{x_1^2+x_2^2}}, \qquad 0 \le y_2 < 1.$$

But $y_1 = \sqrt{x_1^2+x_2^2}$ and, therefore,

$$f_{\mathbf{X}}(\mathbf{x}) = \frac{1}{2\pi}\exp\{-\frac{x_1^2+x_2^2}{2}\} = f_{X_1}(x_1)f_{X_2}(x_2).$$

Thus, X_1 and X_2 are jointly Gaussian and statistically independent.

1.17 From the density (1.49), we obtain the distribution

$$F_Y(y) = \begin{cases} \displaystyle\int_0^y f_Y(u)du = 1 - e^{-y^2/2}, & y \ge 0 \\ 0, & y < 0. \end{cases}$$

1.18 Observe that the inverse function $g^{-1}(\cdot)$ does not exist. However, it can be seen by inspection that

$$f_Y(y) = \begin{cases} \int_{-\infty}^{-1} f_X(x)dx \cdot \delta(y+1), & -\infty < y <= -1 \\ f_X(y), & -1 < y < 1 \\ \int_{1}^{\infty} f_X(x)dx \cdot \delta(y-1), & 1 \le y < \infty. \end{cases}$$

1.19 *a*) Since $0 \le Y \le 1$ and $dg(x)/dx = f_X(x)$, then it follows from (1.36) that

$$f_Y(y) = \frac{f_X(x)}{|dg(x)/dx|} = 1, \qquad 0 \le y \le 1,$$

which is the desired density.

b) From (1.36), we have

$$f_Z(z) = f_Y(h^{-1}(z))\left|\frac{dh^{-1}(z)}{dz}\right| = 1\frac{dF_W(z)}{dz} = f_W(z),$$

which is the desired density.

c) Let $k(\cdot)$ be the composition of $F_W^{-1}(\cdot)$ and $F_X(\cdot)$. Then $Z = F_W^{-1}(F_X(X))$, and it follows from parts *a* and *b* that $F_Z(\cdot) = F_W(\cdot)$, as desired.

1.20 *a*) The probability density for the uniformly quantized random variable Y is given by

$$f_Y(y) = \sum_{n=-\infty}^{\infty} \int_{n-1/2}^{n+1/2} f_X(x)dx \ \delta(y-n).$$

b) The values of the quantization levels $\{x_n\}$ are the solutions to the M equations,

$$\int_{x_n}^{x_{n+1}} f_X(x)dx = \frac{P}{m}, \qquad n = 1, \ldots m,$$

where

$$P \triangleq \int_{x_1}^{x_{m+1}} f_X(x)dx.$$

c) From part *b*, for $x_1 < 0$ and $x_{m+1} > 0$ given, the quantization levels $\{x_n\}$ that yield a uniform probability distribution for Y are found to be

(i) $\quad x_{n+1} = \frac{1}{a}\ln[\frac{2P}{m} + e^{ax_n}], \qquad x_{n+1} < 0$

(ii) $x_{n+1} = \frac{-1}{a}\ln[2 - e^{ax_n} - \frac{2P}{m}], \qquad x_n < 0 < x_{n+1}$

(iii) $x_{n+1} = \frac{-1}{a}\ln[e^{-ax_n} - \frac{2P}{m}], \qquad 0 < x_n < x_{n+1},$

where

$$P = \frac{1}{2}[2 - e^{ax_1} - e^{-ax_{m+1}}].$$

1.21 *a*) In this case, we have from (1.29) the joint probability density

$$f_{XY}(x, y) = f_{Y|X}(y|X=x)f_X(x) = f_Z(y-x)f_X(x).$$

b) In this case, we have from (1.29) and exercise 1 in chapter 2 or (1.39) the joint density

$$f_{XY}(x, y) = f_{Y|X}(y|X=x)f_X(x) = |x|f_Z(xy)f_X(x).$$

1.22 For the half-wave rectifier, we have the distribution

$$\begin{aligned} F_Y(y) &= P[\{Y<y\} \cap \{(X<0) \cup (X \geq 0)\}] \\ &= P[(Y<y) \cap (X<0)] + P[(Y<y) \cap (X \geq 0)] \\ &= P[Y<y|X<0]P[X<0] + P[Y<y|X\geq 0]P[X \geq 0] \\ &= F_{Y|X<0}(y)P + F_{Y|X\geq 0}(y)(1-P), \end{aligned}$$

where

$$P \triangleq P[X<0].$$

Thus, we have the density

$$f_Y(y) = f_{Y|X<0}(y)P + f_{Y|X\geq 0}(y)(1-P).$$

But, if $X < 0$, then $Y = 0$; therefore,

$$f_{Y|X<0}(y) = \delta(y).$$

Also, if $X \geq 0$, then $Y = X$; therefore (cf. exercise 1.8),

$$f_{Y|X\geq 0}(y) = \frac{f_X(y)u(y)}{1-P},$$

where

$$u(y) = \begin{cases} 1, & y > 0 \\ 0, & y \leq 0. \end{cases}$$

Hence, the density is given by

$$f_Y(y) = P\,\delta(y) + f_X(y)u(y),$$

where

$$P \triangleq \int_{-\infty}^{0} f_X(x)dx.$$

1.23 From the definition of conditional probability density, we have

$$P \triangleq Prob\,[(X,Y)\in A\,|(X,Y)\in B\,] = \frac{Prob\,[(X,Y)\in A,\,(X,Y)\in B\,]}{Prob\,[(X,Y)\in B\,]}$$

$$= \begin{cases} \dfrac{Prob\,[(X,Y)\in A\,]}{Prob\,[(X,Y)\in B\,]}, & A \subseteq B \\ 0, & A \subseteq \bar{B}. \end{cases}$$

Let $A = \{(X,Y): x \leq X < x + \varepsilon,\ y \leq Y < y + \varepsilon\}$. Then, with $\varepsilon \to 0$, we obtain

$$P = f_{XY\,|(X,Y)\in B}(x,y) = \begin{cases} \dfrac{f_{XY}(x,y)}{\iint\limits_B f_{XY}(u,v)dudv}, & (x,y) \in B \\ 0, & (x,y) \notin B. \end{cases}$$

Finally, let B be given by

$$B = \{(u,v): u^2 + v^2 < a^2\},$$

and let $X = X_1$ and $Y = X_2$ be zero-mean, unity variance, independent, and Gaussian. Then by a change of variables of integration, it can be shown that

$$\iint\limits_B f_{XY}(u,v)dudv = \iint\limits_{u^2+v^2<a^2} \frac{1}{2\pi} e^{-(u^2+v^2)}dudv = 1 - e^{-a^2/2}.$$

Therefore, the conditional joint probability density is given by

$$f_{X_1X_2|X_1^2+X_2^2<a^2}(x_1,x_2) = \begin{cases} \dfrac{1}{2\pi}\,\dfrac{e^{-(x_1^2+x_2^2)/2}}{1 - e^{-a^2/2}}, & x_1^2 + x_2^2 < a^2 \\ 0, & otherwise. \end{cases}$$

1.24 Let $f_{XY}(x, y) = f_X(x) f_Y(y)$. Then, from (1.39), we have

$$f_{UV}(u, v) = f_{XY}(x, y) \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$= f_X(g^{-1}(u)) f_Y(h^{-1}(v)) \left[\frac{\partial g^{-1}(u)}{\partial u} \frac{\partial h^{-1}(v)}{\partial v} - 0 \right]$$

$$= f_U(u) f_V(v).$$

Thus, U and V are statistically independent as well.

1.25 Using the definition of conditional probability density, we have in this case

$$f_{X|Y}(x | Y = y) = \frac{f_{Y|X}(y | X = x) f_X(x)}{f_Y(y)}.$$

We also have

$$f_{Y|X}(y | X = x) = \delta(y - x^2).$$

The probability distribution function for Y is given by

$$F_Y(y) = Prob\{Y < y\} = Prob\{X^2 < y\}$$

$$= Prob\{[X < \sqrt{y} \text{ and } X \geq 0] \text{ or } [-X < \sqrt{y} \text{ and } X < 0]\}$$

$$= Prob\{0 \leq X < \sqrt{y}\} + Prob\{0 > X > -\sqrt{y}\}$$

$$= \int_0^{\sqrt{y}} f_X(z) dz + \int_{-\sqrt{y}}^{0} f_X(z) dz.$$

Thus,

$$f_Y(y) = \frac{1}{2\sqrt{y}} [f_X(\sqrt{y}) + f_X(-\sqrt{y})]$$

and

$$f_{X|Y}(x | Y = y) = 2\sqrt{y} \frac{\delta(y - x^2) f_X(x)}{f_X(\sqrt{y}) + f_X(-\sqrt{y})}.$$

But,

$$\delta(y - x^2) = \delta(x - \sqrt{y}\,) + \delta(x + \sqrt{y}\,)$$

and

$$\delta(x - a)f_X(x) = \delta(x - a)f_X(a).$$

Hence, the desired conditional density is given by

$$f_{X|Y}(x|Y = y) = 2\sqrt{y}\,\frac{\delta(x - \sqrt{y}\,)f_X(\sqrt{y}\,) + \delta(x + \sqrt{y}\,)f_X(-\sqrt{y}\,)}{f_X(\sqrt{y}\,) + f_X(-\sqrt{y}\,)}.$$

Chapter 2

Expectation

2.1 *a*) We have the uniform probability density

$$f_X(x) = \begin{cases} \dfrac{1}{b-a}, & a \le x < b \\ 0, & \textit{otherwise.} \end{cases}$$

Therefore, the mean and variance are given by

$$E\{X\} = \int_a^b \frac{x}{b-a}dx = \frac{b+a}{2},$$

$$E\{X^2\} = \int_a^b \frac{x^2}{b-a}dx = \frac{b^3-a^3}{3(b-a)} = \frac{a^2+ab+b^2}{3},$$

$$Var\{X\} = E\{X^2\} - E^2\{X\} = \frac{(b-a)^2}{12}.$$

b) For this uniform probability distribution, we have

$$E\{Y\} = \frac{1}{n}\sum_{i=1}^{n} i\Delta = \frac{(n+1)}{2}\Delta.$$

$$Var\{Y\} = E\{Y^2\} - E^2\{Y\} = \frac{1}{n}\sum_{i=1}^{n}(i\Delta)^2 - \frac{(n+1)^2}{4}\Delta^2$$

$$= \frac{n(n+1)(2n+1)}{6n}\Delta^2 - \frac{(n+1)^2}{4}\Delta^2 = \frac{n^2-1}{12}\Delta^2.$$

c) For $b - a = n\Delta$, part *a* yields $Var\{X\} = \dfrac{n^2}{12}\Delta^2 \simeq Var\{Y\}$ for $n \gg 1$.

2.2 *a*) For the specified probability distribution, the mean is given by

$$E\{X\} = \sum_{n=0}^{\infty} nP(X=n) = (1-\gamma)\sum_{n=0}^{\infty} n\gamma^n.$$

But, since $\dfrac{d}{d\gamma}\gamma^n = n\gamma^{n-1}$, then

$$\sum_{n=0}^{\infty} n\gamma^n = \gamma\frac{d}{d\gamma}\sum_{n=0}^{\infty}\gamma^n = \gamma\frac{d}{d\gamma}\frac{1}{(1-\gamma)} = \frac{\gamma}{(1-\gamma)^2}.$$

Thus,

$$E\{X\} = \frac{\gamma}{1-\gamma}.$$

Similarly, the mean squared value can be obtained as follows:

$$E\{X^2\} = \sum_{n=0}^{\infty} n^2(1-\gamma)\gamma^n = (1-\gamma)\left[\gamma^2\frac{d^2}{d\gamma^2}\sum_{n=0}^{\infty}\gamma^n + \sum_{n=0}^{\infty} n\gamma^n\right]$$

$$= (1-\gamma)\left[\frac{2\gamma^2}{(1-\gamma)^3} + \frac{\gamma}{(1-\gamma)^2}\right] = \frac{2\gamma^2}{(1-\gamma)^2} + \frac{\gamma}{(1-\gamma)}.$$

Therefore, the variance is given by

$$Var\{X\} = E\{X^2\} - E^2\{X\} = \frac{\gamma^2}{(1-\gamma)^2} + \frac{\gamma}{1-\gamma} = \frac{\gamma}{(1-\gamma)^2}.$$

b) For this exponential density, we have the mean

$$E\{Y\} = \int_0^{\infty} ybe^{-by}\,dy = -\int_0^{\infty} yd(e^{-by}) = -ye^{-by}\Big|_0^{\infty} + \int_0^{\infty} e^{-by}\,dy = \frac{1}{b}$$

and the mean square

$$E\{Y^2\} = \int_0^{\infty} y^2be^{-by}\,dy = -\int_0^{\infty} y^2d(e^{-by}) = -y^2e^{-by}\Big|_0^{\infty} + 2\int_0^{\infty} ye^{-by}\,dy = \frac{2}{b^2}.$$

Therefore, the variance is given by

$$Var\{Y\} = E\{Y^2\} - E^2\{Y\} = \frac{1}{b^2}.$$

2.3 For this scaled random variable Y, we have the mean

$$m_Y = \int_{-\infty}^{\infty} yf_Y(y)dy = \int_{-\infty}^{\infty} \frac{y}{|a|}f_X(\frac{y}{a})dy$$

$$= \begin{cases} \displaystyle\int_{-\infty}^{\infty} \frac{y}{a}f_X(\frac{y}{a})dy, & a > 0 \\ \displaystyle-\int_{-\infty}^{\infty} \frac{y}{a}f_X(\frac{y}{a})dy, & a < 0 \end{cases} = \begin{cases} \displaystyle a\int_{-\infty}^{\infty} uf_X(u)du, & a > 0 \\ \displaystyle -a\int_{\infty}^{-\infty} uf_X(u)du, & a < 0 \end{cases} = am_X;$$

also, $m_Y = E\{aX\} = aE\{X\} = am_X$. In addition, we have the mean square

$$E\{Y^2\} = \int_{-\infty}^{\infty} y^2f_Y(y)dy = \int_{-\infty}^{\infty} \frac{y^2}{|a|}f_X(\frac{y}{a})dy$$

$$= \left\{ \begin{array}{ll} a^2 \int\limits_{-\infty}^{\infty} u^2 f_X(u)du, & a > 0 \\ -a^2 \int\limits_{\infty}^{-\infty} u^2 f_X(u)du, & a < 0 \end{array} \right\} = a^2 E\{X^2\};$$

also, $E\{Y^2\} = E\{(aX)^2\} = E\{a^2X^2\} = a^2E\{X^2\}$. Therefore, the variance is given by

$$\sigma_Y^2 = E\{Y^2\} - m_Y^2 = a^2(E\{X^2\} - m_X^2) = a^2\sigma_X^2.$$

2.4 Using $Y = g(X)$ and (1.36) yields the mean

$$E\{Y\} = \int\limits_{-\infty}^{\infty} y f_Y(y)dy = \int\limits_{y=-\infty}^{\infty} g(x) \frac{f_X(x)}{|dg(x)/dx|} \frac{dg(x)}{dx} dx$$

$$= \left\{ \begin{array}{ll} \int\limits_{x=-\infty}^{\infty} g(x) f_X(x)dx, & \dfrac{dg}{dx} > 0 \\ -\int\limits_{x=\infty}^{-\infty} g(x) f_X(x)dx, & \dfrac{dg}{dx} < 0 \end{array} \right.$$

$$= \int\limits_{-\infty}^{\infty} g(x) f_X(x)dx \triangleq E\{g(X)\},$$

which is the desired result (2.8).

2.5 *a*) The relationship (2.17) can be verified as follows:

$$\sigma_X^2 = E\{(X - m_X)^2\} = E\{X^2 - 2m_X X + m_X^2\}$$
$$= E\{X^2\} - 2m_X E\{X\} + m_X^2 = E\{X^2\} - m_X^2.$$

b) The relationship (2.22) can be verified as follows:

$$K_{XY} = E\{(X - m_X)(Y - m_Y)\}$$
$$= E\{XY\} - m_X E\{Y\} - m_Y E\{X\} + m_X m_Y$$
$$= R_{XY} - m_X m_Y.$$

2.6 *a*) For the Gaussian density (1.23), the mean is given by

$$m_X = E\{X\} = \int_{-\infty}^{\infty} \frac{x}{\alpha\sqrt{2\pi}} \exp\{-\frac{1}{2}(\frac{x-\beta}{\alpha})^2\}dx = \int_{-\infty}^{\infty} \frac{\alpha y+\beta}{\alpha\sqrt{2\pi}} \exp\{-\frac{y^2}{2}\}\alpha dy$$

$$= \beta \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\{-\frac{y^2}{2}\}dy + \alpha \int_{-\infty}^{\infty} \frac{y}{\sqrt{2\pi}} \exp\{-\frac{y^2}{2}\}dy.$$

The first integral is unity as shown in exercise 1.12*a*. The second integral is zero because its integrand is the product of an even function and an odd function and is, therefore, an odd function; thus,

$$m_X = \beta 1 + \alpha 0 = \beta.$$

The variance for this density is given by

$$\sigma_X^2 = E\{X^2\} - m_X^2 = \int_{-\infty}^{\infty} \frac{x^2}{\alpha\sqrt{2\pi}} \exp\{-\frac{1}{2}(\frac{x-\beta}{\alpha})^2\}dx - \beta^2$$

$$= \int_{-\infty}^{\infty} \frac{(\alpha y+\beta)^2}{\alpha\sqrt{2\pi}} \exp\{-\frac{y^2}{2}\}\alpha dy - \beta^2$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (\alpha^2 y^2 + 2\alpha\beta y + \beta^2) \exp\{-\frac{y^2}{2}\}dy - \beta^2$$

$$= \alpha^2 1 + 2\alpha\beta 0 + \beta^2 1 - \beta^2 = \alpha^2.$$

The coefficients 0 and 1 in the second and third terms are obtained as in the first part. The coefficient 1 in the first term is obtained from integration by parts:

$$\int_{-\infty}^{\infty} y^2 \exp\{-\frac{y^2}{2}\}dy = \sqrt{2\pi}.$$

b) The correlation coefficient is defined by

$$\rho \triangleq \frac{K_{XY}}{\sigma_X \sigma_Y} = \frac{R_{XY} - m_X m_Y}{\sigma_X \sigma_Y},$$

where

$$R_{XY} = E\{XY\} = E\{E\{X|Y\}Y\}.$$

From exercise 1.12*d*, we also have for these jointly Gaussian variables

$$f_{X|Y}(x|Y=y) = \frac{1}{\alpha''\sqrt{2\pi}} \exp\{-\frac{1}{2}(\frac{x-\beta''}{\alpha''})^2\},$$

where

$$\beta'' = m_X + \frac{\gamma(y - m_Y)\sigma_X}{\sigma_Y} \quad \text{and} \quad \alpha'' = \sigma_X\sqrt{1-\gamma^2}.$$

Therefore, $E\{X|Y\} = \beta''$ and

$$R_{XY} = E\{E\{X|Y\}Y\} = E\{(m_X + \frac{\gamma(Y - m_Y)\sigma_X}{\sigma_Y})Y\}$$

$$= (m_X - \gamma m_Y\frac{\sigma_X}{\sigma_Y})m_Y + \gamma\frac{\sigma_X}{\sigma_Y}(\sigma_Y^2 + m_Y^2)$$

$$= \gamma\sigma_X\sigma_Y + m_X m_Y.$$

Consequently, $\rho = \gamma$.

2.7 The probability density for the quantization error $E = X - \tilde{X}$ is given by

$$f_E(e) = \begin{cases} 1/\Delta, & |e| \leq \Delta/2 \\ 0, & |e| > \Delta/2, \end{cases}$$

where $\Delta = 30/(n-1)$. Therefore (see exercise 2.1a), the error variance is given by

$$\sigma_E^2 = \frac{[30/(n-1)]^2}{12}.$$

Thus, for $\sigma_E^2 \leq 0.3^2$, we require $n \geq 30$.

2.8 a) Fourier transforming the given characteristic function yields

$$f_X(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-a|\omega|}e^{-i\omega x}\,d\omega = \frac{1}{2\pi}\left[\int_{-\infty}^{0} e^{a\omega}e^{-i\omega x}\,d\omega + \int_{0}^{\infty} e^{-a\omega}e^{-i\omega x}\,d\omega\right]$$

$$= \frac{1}{2\pi}\left[\frac{1}{a-ix} + \frac{1}{a+ix}\right] = \frac{a/\pi}{x^2+a^2},$$

which is the Cauchy probability density.

b) Since $f_X(x)$ is even, then $E\{X\} = 0$ and, therefore, the variance is given by

$$Var\{X\} = E\{X^2\} = \frac{a}{\pi}\int_{-\infty}^{\infty}\frac{x^2}{x^2+a^2}dx = \frac{2a}{\pi}\left[x\Big|_0^{\infty} - a\tan^{-1}\frac{x}{a}\Big|_0^{\infty}\right] = \infty\ ;$$

that is, the variance does not exit.

2.9 Since the Gaussian density is given by

$$f_X(x) = \frac{1}{\alpha\sqrt{2\pi}}\exp\{-\frac{1}{2}(\frac{x-\beta}{\alpha})^2\},$$

then it can be shown (by Fourier transforming) that the characteristic function is given by

$$\Phi_X(\omega) = \exp\{i\beta\omega - \frac{1}{2}\alpha^2\omega^2\}.$$

From (2.14), we then obtain the following moments:

$$E\{X\} = \frac{1}{i}\frac{d\Phi_X(\omega)}{d\omega}\bigg|_{\omega=0} = \beta,$$

$$E\{X^2\} = \frac{1}{i^2}\frac{d^2\Phi_X(\omega)}{d\omega^2}\bigg|_{\omega=0} = \beta^2 + \alpha^2,$$

$$E\{X^3\} = \frac{1}{i^3}\frac{d^3\Phi_X(\omega)}{d\omega^3}\bigg|_{\omega=0} = \beta^3 + 3\beta\alpha^3,$$

$$E\{X^4\} = \frac{1}{i^4}\frac{d^4\Phi_X(\omega)}{d\omega^4}\bigg|_{\omega=0} = 3\alpha^4 + \beta^4 + 6\alpha^2\beta^2.$$

2.10 From (2.13), we have

$$\frac{d^n\Phi_X(\omega)}{d\omega^n} = \int_{-\infty}^{\infty} \frac{d^n}{d\omega^n}e^{i\omega x}f_X(x)dx = \int_{-\infty}^{\infty}(ix)^n e^{i\omega x}f_X(x)dx.$$

Thus,

$$\frac{1}{i^n}\frac{d^n\Phi_X(\omega)}{d\omega^n}\bigg|_{\omega=0} = \int_{-\infty}^{\infty}x^n f_X(x)dx = E\{X^n\},$$

which verifies (2.14).

2.11 From (1.32), we have the convolution

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(z-u)f_Y(u)du.$$

From (2.13), we therefore have the characteristic function

$$\Phi_Z(\omega) = \int_{-\infty}^{\infty} e^{i\omega z}f_Z(z)dz = \iint_{-\infty}^{\infty} e^{i\omega z}f_X(z-u)dz\ f_Y(u)du$$

$$= \int_{-\infty}^{\infty}\Phi_X(\omega)e^{i\omega u}f_Y(u)du = \Phi_X(\omega)\Phi_Y(\omega),$$

which proves the convolution theorem.

2.12 We have the random variable

$$Y = \frac{1}{n}\sum_{k=1}^{n} X_k^2 - \frac{2}{n^2}\sum_{k,j=1}^{n} X_k X_j + \frac{1}{n^2}\sum_{i,j=1}^{n} X_i X_j = \frac{1}{n}\sum_{k=1}^{n} X_k^2 - \frac{1}{n^2}\sum_{i,j=1}^{n} X_i X_j .$$

Therefore, for i.i.d. zero-mean random variable $\{X_i\}$, we have the mean

$$E\{Y\} = \frac{1}{n}\sum_{k=1}^{n}\sigma_X^2 - \frac{1}{n^2}\sum_{i=j=1}^{n}\sigma_X^2 = \sigma_X^2 - \frac{\sigma_X^2}{n} = \frac{n-1}{n}\sigma_X^2 \to \sigma_X^2, \text{ as } n \to \infty.$$

Also, since the squared random variable Y^2 can be expressed as

$$Y^2 = \frac{1}{n^2}\sum_{k,l=1}^{n} X_k^2 X_l^2 + \frac{1}{n^4}\sum_{k,l,i,j=1}^{n} X_k X_l X_i X_j - \frac{2}{n^3}\sum_{k,i,j=1}^{n} X_k^2 X_i X_j ,$$

then using the hint yields (for Gaussian variables $\{X_i\}$) the mean square

$$\begin{aligned} E\{Y^2\} &= \frac{1}{n^2}\sum_{k,l=1}^{n} [\, E\{X_k^2\}E\{X_l^2\} + 2E^2\{X_k X_l\}\,] \\ &+ \frac{1}{n^4}\sum_{k,l,i,j=1}^{n} [E\{X_k X_l\}E\{X_i X_j\} + E\{X_k X_i\}E\{X_l X_j\} + E\{X_k X_j\}E\{X_l X_i\}] \\ &- \frac{2}{n^3}\sum_{k,i,j=1}^{n} [E\{X_k^2\}E\{X_i X_j\} + 2E\{X_k X_i\}E\{X_k X_j\}] \\ &= (\sigma_X^4 + \frac{2}{n}\sigma_X^4) + \frac{1}{n^4}(3n^2\sigma_X^4) - \frac{2}{n^3}(n^2\sigma_X^4 + 2n\,\sigma_X^4) \\ &= \frac{(n^2-1)}{n^2}\sigma_X^4 . \end{aligned}$$

Thus, the variance is given by

$$Var\{Y\} = E\{Y^2\} - E^2\{Y\} = \frac{2(n-1)}{n^2}\sigma_X^4 \to \frac{2\sigma_X^4}{n} \to 0 \text{ as } n \to \infty.$$

2.13 *a*) Let $M = N_1 + N_2$, where N_1 and N_2 are independent and have the probability mass functions

$$P_{N_1}(n_1) = \frac{\lambda_1^{n_1}}{n_1!}e^{-\lambda_1} \quad \text{and} \quad P_{N_2}(n_2) = \frac{\lambda_2^{n_2}}{n_2!}e^{-\lambda_2}, \quad n_1, n_2 = 0, 1, 2, \ldots .$$

Then, the probability mass function for M is given by

$$P_M(m) = \sum_{n_1=0}^{m} P_{N_1}(n_1)P_{N_2}(m-n_1) = \sum_{n_1=0}^{m} \frac{\lambda_1^{n_1}}{n_1!}\frac{\lambda_2^{m-n_1}}{(m-n_1)!}e^{-(\lambda_1+\lambda_2)}$$

$$= \frac{e^{-(\lambda_1+\lambda_2)}}{m!} \sum_{n_1=0}^{m} \binom{m}{n_1} \lambda_1^{n_1} \lambda_2^{m-n_1} = \frac{(\lambda_1+\lambda_2)^m}{m!} e^{-(\lambda_1+\lambda_2)},$$

which is a Poisson distribution with parameter $\lambda = \lambda_1 + \lambda_2$. Thus, by induction, for $M = \sum_{i=1}^{m} N_i$, we have

$$P_M(k) = \frac{\left(\sum_{i=1}^{m} \lambda_i\right)^k}{k!} \exp\{-\sum_{i=1}^{m} \lambda_i\}.$$

b) Using the identity $e^{\lambda} = \sum_{k=0}^{\infty} \frac{\lambda^k}{k!}$ yields

$$Mean = \sum_{n=0}^{\infty} nP(n) = \sum_{n=1}^{\infty} \frac{\lambda^n}{(n-1)!} e^{-\lambda} = \lambda \sum_{m=0}^{\infty} \frac{\lambda^m}{m!} e^{-\lambda} = \lambda$$

and

$$\begin{aligned} Variance &= \sum_{n=0}^{\infty} n^2 P(n) - \lambda^2 = \sum_{n=1}^{\infty} \frac{n}{(n-1)!} \lambda^n e^{-\lambda} - \lambda^2 \\ &= \sum_{n=1}^{\infty} \frac{n-1}{(n-1)!} \lambda^n e^{-\lambda} + \sum_{n=1}^{\infty} \frac{\lambda^n}{(n-1)!} e^{-\lambda} - \lambda^2 \\ &= \sum_{n=2}^{\infty} \frac{\lambda^n}{(n-2)!} e^{-\lambda} + \lambda \sum_{m=0}^{\infty} \frac{\lambda^m}{m!} e^{-\lambda} - \lambda^2 \\ &= \lambda^2 \sum_{m=0}^{\infty} \frac{\lambda^m}{m!} e^{-\lambda} + \lambda - \lambda^2 = \lambda. \end{aligned}$$

2.14 Let $w(x, y) = f_{XY}(x, y)$, $h(x, y) = x$, and $g(x, y) = y$. Then the Cauchy-Schwarz inequality (2.65) yields

$$|E\{XY\}|^2 \le E\{X^2\}E\{Y^2\}.$$

This must also be true for $X' = X - m_X$ and $Y' = Y - m_Y$. Therefore,

$$|Cov\{X, Y\}|^2 \le Var\{X\}Var\{Y\},$$

from which we obtain $|\rho| \le 1$.

2.15 *a*) Let $w(x, y) = f_{XY}(x, y)$, $g(x, y) = x - m_X$, and $h(x, y) = y - m_Y$. Then the triangle inequality (2.66) yields

$$E\{[X + Y - (m_X + m_Y)]^2\}^{1/2} \le E\{(X - m_X)^2\}^{1/2} + E\{(Y - m_Y)^2\}^{1/2},$$

from which we obtain

$$\sigma_{X+Y} \le \sigma_X + \sigma_Y.$$

For $W = X + Y + Z$, this yields

$$\sigma_W \le \sigma_{X+Y} + \sigma_Z \le \sigma_X + \sigma_Y + \sigma_Z,$$

and similarly, for the sum of any finite number of random variables, we obtain the sum of standard deviations.

b) We have

$$E\{(X+Y)^2\} - E^2\{X+Y\}$$
$$= E\{X^2\} + E\{Y^2\} + 2E\{XY\} - E^2\{X\} - E^2\{Y\} - 2E\{X\}E\{Y\}.$$

Since $E\{XY\} = E\{X\}E\{Y\}$ for statistically independent random variables, then this reduces to

$$\sigma_{X+Y}^2 = \sigma_X^2 + \sigma_Y^2.$$

For $W = X + Y + Z$, this yields

$$\sigma_W^2 = \sigma_{X+Y}^2 + \sigma_Z^2 = \sigma_X^2 + \sigma_Y^2 + \sigma_Z^2,$$

and similarly, for the sum of any finite number of random variables, we obtain the sum of variances.

2.16 Contours of constant elevation are always ellipses for jointly Gaussian variables. When $|\rho| \to 1$, the contours collapse to a line. When $|\rho| \to 0$ and $\sigma_X = \sigma_Y$, the contours are circles.

2.17 From the description in the exercise, we have the joint density

$$f_{XY}(x, y) = \frac{1}{ab}$$

with x and y bounded by the four line-segments

$$y = x + \frac{b}{\sqrt{2}}, \quad \text{for} \quad -\frac{(a+b)}{2\sqrt{2}} \le x \le \frac{(a-b)}{2\sqrt{2}}$$

$$y = x - \frac{b}{\sqrt{2}}, \quad \text{for} \quad -\frac{(a-b)}{2\sqrt{2}} \le x \le \frac{(a+b)}{2\sqrt{2}}$$

$$y = -x + \frac{a}{\sqrt{2}}, \quad \text{for} \quad \frac{(a-b)}{2\sqrt{2}} \le x \le \frac{(a+b)}{2\sqrt{2}}$$

$$y = -x - \frac{a}{\sqrt{2}}, \quad \text{for} \quad -\frac{(a+b)}{2\sqrt{2}} \le x \le -\frac{(a-b)}{2\sqrt{2}}.$$

The marginal density for X is therefore

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x, y)dy$$

$$= \begin{cases} \dfrac{1}{ab}\displaystyle\int_{-x-a/\sqrt{2}}^{x+b/\sqrt{2}} dy = \dfrac{1}{ab}(2x + \dfrac{a+b}{\sqrt{2}}), & -\dfrac{a+b}{2\sqrt{2}} \le x < -\dfrac{a-b}{2\sqrt{2}} \\ \dfrac{1}{ab}\displaystyle\int_{x-b/\sqrt{2}}^{x+b/\sqrt{2}} dy = \dfrac{1}{ab}\dfrac{2b}{\sqrt{2}}, & -\dfrac{a-b}{2\sqrt{2}} \le x < \dfrac{a-b}{2\sqrt{2}} \\ \dfrac{1}{ab}\displaystyle\int_{x-b/\sqrt{2}}^{-x+a/\sqrt{2}} dy = \dfrac{1}{ab}(-2x + \dfrac{a+b}{\sqrt{2}}), & \dfrac{a-b}{2\sqrt{2}} \le x \le \dfrac{a+b}{2\sqrt{2}}. \end{cases}$$

Replacing x by y in the above equation yields the marginal density for Y. From f_X and f_Y we can show that $m_X = m_Y = 0$ and $\sigma_X = \sigma_Y = \dfrac{a^2+b^2}{24}$. Also, the correlation is given by

$$R_{XY} = \iint_{-\infty}^{\infty} xyf_{XY}(x, y)dxdy$$

$$= \frac{1}{ab}\int_{-\frac{a+b}{2\sqrt{2}}}^{-\frac{a-b}{2\sqrt{2}}} y \int_{-y-\frac{a}{\sqrt{2}}}^{y+\frac{b}{\sqrt{2}}} x\ dxdy + \frac{1}{ab}\int_{-\frac{a-b}{2\sqrt{2}}}^{\frac{a-b}{2\sqrt{2}}} y \int_{y-\frac{b}{\sqrt{2}}}^{y+\frac{b}{\sqrt{2}}} x\ dxdy + \frac{1}{ab}\int_{\frac{a-b}{2\sqrt{2}}}^{\frac{a+b}{2\sqrt{2}}} y \int_{y-\frac{b}{\sqrt{2}}}^{-y+\frac{a}{\sqrt{2}}} x\ dxdy$$

$$= \frac{1}{ab}\int_{-\frac{a+b}{2\sqrt{2}}}^{-\frac{a-b}{2\sqrt{2}}} \frac{y}{2}[(y+\frac{b}{\sqrt{2}})^2 - (y+\frac{a}{\sqrt{2}})^2]dy + \frac{1}{ab}\int_{-\frac{a-b}{2\sqrt{2}}}^{\frac{a-b}{2\sqrt{2}}} \frac{y}{2}[(y+\frac{b}{\sqrt{2}})^2 - (y-\frac{b}{\sqrt{2}})^2]dy$$

$$+ \frac{1}{ab}\int_{\frac{a-b}{2\sqrt{2}}}^{\frac{a+b}{2\sqrt{2}}} \frac{y}{2}[(y-\frac{a}{\sqrt{2}})^2 - (y-\frac{b}{\sqrt{2}})^2]dy$$

$$= \frac{1}{ab}\int_{-\frac{a+b}{2\sqrt{2}}}^{-\frac{a-b}{2\sqrt{2}}} [\frac{b^2-a^2}{2}y + (\frac{2b}{\sqrt{2}} - \frac{2a}{\sqrt{2}})y^2]dy + \frac{4}{\sqrt{2}a}\int_0^{\frac{a-b}{2\sqrt{2}}} y^2dy = \frac{a^2-b^2}{24}.$$

Therefore, the correlation coefficient is given by

$$\rho = \frac{K_{XY}}{\sigma_X\sigma_Y} = \frac{R_{XY} - m_X m_Y}{\sigma_X\sigma_Y} = \frac{a^2-b^2}{a^2+b^2}.$$

When $a = b$, then $\rho = 0$; when $b = 0$, then $\rho = 1$; and when $a = 0$, then $\rho = -1$.

2.18 *a*) For $N = X - aZ$, we have

$$E\{N^2\} = a^2E\{Z^2\} - 2aE\{XZ\} + E\{X^2\}.$$

Therefore,

$$\frac{dE\{N^2\}}{da} = 2aE\{Z^2\} - 2E\{XZ\} = 0 \implies a = \frac{E\{XZ\}}{E\{Z^2\}}$$

and, consequently,

$$E\{NZ\} = E\{(X - aZ)Z\} = E\{XZ\} - aE\{Z^2\} = 0$$

when $E\{N^2\}$ is minimum.

b) We have the two variables

$$X = a_1Z + N_1 \quad \text{and} \quad Y = a_2Z + N_2,$$

for which $E\{N_1Z\} = E\{N_2Z\} = 0$. Thus,

$$E\{XY\} = a_1a_2E\{Z^2\} + E\{N_1N_2\}.$$

Let X and Y be defined such that

$$E\{N_1N_2\} = -a_1a_2E\{Z^2\}.$$

Then X and Y are orthogonal. We can show the same for $X - m_X$ and $Y - m_Y$, in which case X and Y would be uncorrelated.

2.19 *a*) If $f_{XY}(x, y) = f_X(x)f_Y(y)$, then

$$K_{XY} = E\{XY\} - m_Xm_Y = \iint_{-\infty}^{\infty} xyf_X(x)f_Y(y)dxdy - m_Xm_Y = 0.$$

b) We have the correlation

$$R_{XY} = E\{XY\} = E\{X^3\} = \int_{-\infty}^{\infty} x^3f_X(x)dx = 0$$

since the integrand is an odd function. Also, we have

$$m_X = E\{X\} = \int_{-\infty}^{\infty} xf_X(x)dx = 0.$$

Thus,

$$K_{XY} = R_{XY} - m_Xm_Y = 0,$$

and X and Y are therefore uncorrelated. But X and $Y = X^2$ are obviously not statistically independent.

2.20 We have the expected sum of squares

$$E\{[\sum_{i=1}^{n} c_i(X_i - m_{X_i})]^2\} = E\{\sum_{i,j=1}^{n} c_i c_j (X_i - m_{X_i})(X_j - m_{X_j})\}$$

$$= \sum_{i,j=1}^{n} c_i c_j K_{X_i X_j} = \mathbf{c}^T \mathbf{K_X} \mathbf{c},$$

where the covariance matrix $\mathbf{K_X}$ is defined to have ij-th element $K_{X_i X_j}$ and $\mathbf{c}$ is defined to have i-th element c_i. Since

$$\mathbf{c}^T \mathbf{K_X} \mathbf{c} = E\{\sum_{i,j=1}^{n} c_i c_j (X_i - m_{X_i})(X_j - m_{X_j})\}$$

$$= E\{\sum_{i,j=1}^{n} c_i c_j X_i X_j\} - \sum_{i,j=1}^{n} c_i c_j m_{X_i} m_{X_j} = E\{[\sum_{i=1}^{n} c_i X_i]^2\} - (\mathbf{m_X^T c})^2,$$

where $\mathbf{m_X}$ is defined to have i-th element m_{X_i}, then

$$E\{[\sum_{i=1}^{n} c_i X_i]^2\} = \mathbf{c}^T \mathbf{K_X} \mathbf{c} + (\mathbf{m_X^T c})^2.$$

2.21 Since $\mathbf{Y} = [Y_1 \quad Y_2 \ldots Y_n]^T = \mathbf{R}_X^{-1/2}\mathbf{X}$ and $(\mathbf{R}_X^{-1/2})^T = \mathbf{R}_X^{-1/2}$, then

$$E\{\mathbf{YY}^T\} = E\{\mathbf{R}_X^{-1/2}\mathbf{XX}^T\mathbf{R}_X^{-1/2}\} = \mathbf{R}_X^{-1/2}E\{\mathbf{XX}^T\}\mathbf{R}_X^{-1/2}$$

$$= \mathbf{R}_X^{-1/2}\mathbf{R}_X\mathbf{R}_X^{-1/2} = \mathbf{R}_X^{-1/2}\,(\mathbf{R}_X^{1/2}\mathbf{R}_X^{1/2})\,\mathbf{R}_X^{-1/2} = \mathbf{I}$$

and, therefore, $Y_1, Y_2, \ldots, Y_n$ are mutually orthogonal and have unity mean squared values:

$$E\{Y_i Y_j\} = \begin{cases} 1, & i = j \\ 0, & i \neq j. \end{cases}$$

2.22 Let the density functions for the independent Gaussian random variables X and Y be denoted by

$$f_X(x) = \frac{1}{\sigma_X\sqrt{2\pi}}\exp\{-\frac{1}{2}(\frac{x - m_X}{\sigma_X})^2\}$$

and

$$f_Y(y) = \frac{1}{\sigma_Y\sqrt{2\pi}}\exp\{-\frac{1}{2}(\frac{y - m_Y}{\sigma_Y})^2\},$$

respectively, and let the complementary error function be denoted by

$$erfc(z) \triangleq \frac{2}{\sqrt{\pi}} \int_z^{\infty} e^{-w^2} dw.$$

Then,

$$\begin{aligned} P(XY>0) &= P(X>0 \text{ and } Y>0) + P(X<0 \text{ and } Y<0) \\ &= P(X>0)P(Y>0) + P(X<0)P(Y<0) \\ &= P(X>0)P(Y>0) + [1-P(X>0)][1-P(Y>0)] \\ &= 1 + 2P(X>0)P(Y>0) - P(X>0) - P(Y>0) \\ &= 1 + \frac{1}{2} erfc\left(\frac{-m_X}{\sqrt{2}\sigma_X}\right) erfc\left(\frac{-m_Y}{\sqrt{2}\sigma_Y}\right) - \frac{1}{2} erfc\left(\frac{-m_X}{\sqrt{2}\sigma_X}\right) - \frac{1}{2} erfc\left(\frac{-m_Y}{\sqrt{2}\sigma_Y}\right), \end{aligned}$$

since

$$\begin{aligned} P(X>0) &= \int_0^{\infty} \frac{1}{\sigma_X \sqrt{2\pi}} \exp\left\{-\frac{1}{2}\left(\frac{u-m_X}{\sigma_X}\right)^2\right\} du = \int_{-m_X/\sqrt{2}\sigma_X}^{\infty} \frac{1}{\sqrt{\pi}} e^{-w^2} dw \\ &= \frac{1}{2} erfc\left(\frac{-m_X}{\sqrt{2}\sigma_X}\right). \end{aligned}$$

2.23 *a*) From the definition, the *Median* x_0 must satisfy

$$\int_{-\infty}^{x_0} f_X(x)dx = \int_{x_0}^{\infty} f_X(x) = \frac{1}{2}.$$

Also,

$$F_X(x_0) = 1 - F_X(x_0) = \frac{1}{2}.$$

For the Gaussian density, $f_X(x) = \frac{1}{\alpha\sqrt{2\pi}} \exp\{-\frac{1}{2}(\frac{x-\beta}{\alpha})^2\}$, we have the mean value

$$Mean = \int_{-\infty}^{\infty} \frac{x}{\alpha\sqrt{2\pi}} \exp\left\{-\frac{1}{2}\left(\frac{x-\beta}{\alpha}\right)^2\right\} dx = \beta$$

and since

$$\frac{1}{2} = \int_{-\infty}^{x_0} \frac{1}{\alpha\sqrt{2\pi}} \exp\left\{-\frac{1}{2}\left(\frac{x-\beta}{\alpha}\right)^2\right\} dx = \int_{-\infty}^{(x_0-\beta)/\alpha} \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{x^2}{2}\right\} dx,$$

then $x_0 = \beta$. Thus, we have *Mean* $=$ *Median*.

For the Rayleigh density, $f_X(x) = \frac{x}{\alpha^2}\exp\{-\frac{1}{2}(\frac{x}{\alpha})^2\}$ for $x \geq 0$, we have

$$Mean = \int_0^\infty \frac{x^2}{\alpha^2}\exp\{-\frac{1}{2}(\frac{x}{\alpha})^2\}dx = -x\exp\{-\frac{1}{2}(\frac{x}{\alpha})^2\}\Big|_0^\infty$$

$$+\int_0^\infty \exp\{-\frac{1}{2}(\frac{x}{\alpha})^2\}dx = \frac{\alpha\sqrt{2\pi}}{2}$$

and since

$$\frac{1}{2} = \int_0^{x_0} \frac{x}{\alpha^2}\exp\{-\frac{1}{2}(\frac{x}{\alpha})^2\}dx = 1 - \exp\{-\frac{1}{2}(\frac{x_0}{\alpha})^2\},$$

then $x_0 = \alpha\sqrt{2ln\,2}$. Thus, we have $Mean \neq Median$.

Since it can be shown that

$$E\{|X - a|\} = E\{|X - x_0|\} + e,$$

where

$$e \triangleq 2\int_a^{x_0} (x-a)f_X(x)dx \geq 0,$$

then $a = x_0$ yields the minimum of the mean absolute error. In addition, since

$$E\{[X-a]^2\} = E\{|X|^2\} - 2aE\{X\} + a^2,$$

then the derivative with respect to a is zero when $a = E\{X\} = m_X$, and this yields the minimum of the mean squared error.

b) From the definition, the *Mode* x_* must satisfy $\frac{df_X(x)}{dx}\Big|_{x=x_*} = 0$ and, therefore, x_* need not be unique.

For the Gaussian density, we have

$$\frac{df_X(x)}{dx} = -\frac{1}{\alpha\sqrt{2\pi}}\frac{x-\beta}{\alpha}\exp\{-\frac{1}{2}(\frac{x-\beta}{\alpha})^2\} = 0$$

and, therefore, $x_* = \beta$. Thus, we have $Mean = Median = Mode$.

For the Rayleigh density, we have

$$\frac{df_X(x)}{dx} = \frac{1}{\alpha^2}\exp\{-\frac{1}{2}(\frac{x}{\alpha})^2\} - \frac{x^2}{\alpha^4}\exp\{-\frac{1}{2}(\frac{x}{\alpha})^2\} = 0$$

and, therefore, $x_* = \alpha$. Thus, we have $Mean \neq Median \neq Mode \neq Mean$.

In general, if $f_X(\cdot)$ is an even function, then $Mean = Median$; if $f_X(\cdot)$ is also unimodal, then $Mean = Median = Mode$.

2.24 *a*) From (1.24), we obtain the variances $\sigma_A^2 = 1.6 \times 10^{-10}$ V^2 and $\sigma_B^2 = 6.4 \times 10^{-10}$ V^2.

Since $E\{A\} = m_A = E\{B\} = m_B = 0$ and $\rho = 1/2$, then

$$Var\{A+B\} = E\{(A+B)^2\} = E\{A^2\} + E\{B^2\} + 2E\{AB\}$$
$$= \sigma_A^2 + \sigma_A^2 + 2\rho\sigma_A\sigma_B = 11.2 \times 10^{-10} \quad V^2.$$

b) From exercise 1.12*d*, we have the conditional mean

$$E\{B\,|\,A\} = m_B + \rho\frac{\sigma_B}{\sigma_A}(A - m_A) = A = 10^{-5} \quad V.$$

From exercise 1.12*d*, we also have the conditional variance

$$Var\{B\,|\,A\} = \sigma_B^2(1-\rho^2) = 4.8 \times 10^{-10} \quad V^2.$$

2.25 *a*) To verify the formula (2.38) for the joint characteristic function for jointly Gaussian variables, we proceed as follows. The probability density function for a single Gaussian random variable Y is given by

$$f_Y(y) = \frac{1}{\alpha\sqrt{2\pi}}\exp\{-\frac{1}{2}(\frac{x-\beta}{\alpha})^2\}.$$

The characteristic function for Y is given by

$$\Phi_Y(\omega) = E\{e^{iY\omega}\} = \int_{-\infty}^{\infty}\frac{1}{\alpha\sqrt{2\pi}}\exp\{-\frac{1}{2}(\frac{y-\beta}{\alpha})^2\}e^{iy\omega}dy.$$

Using the change of variable $u = (y-\beta)/\alpha$ in the above equation yields

$$\Phi_Y(\omega) = \exp\{i\beta\omega - \frac{1}{2}(\omega\alpha)^2\}\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}\exp\{-\frac{1}{2}(u-i\alpha\omega)^2\}du = \exp\{i\beta\omega - \frac{1}{2}(\omega\alpha)^2\}.$$

Therefore, for $\omega = 1$, we have $\Phi_Y(1) = e^{i\beta - \alpha^2/2}$. Next, we let $Y \triangleq \mathbf{X}^T\boldsymbol{\omega}$, where the elements of the vector $\mathbf{X}$ are jointly Gaussian and Y is, therefore, Gaussian. Then

$$\beta = E\{Y\} = E\{\mathbf{X}^T\boldsymbol{\omega}\} = E\{\mathbf{X}^T\}\boldsymbol{\omega} = \mathbf{m}_{\mathbf{X}}^T\boldsymbol{\omega}$$

and

$$\alpha^2 = Var\{Y\} = E\{(Y-\beta)^2\}$$
$$= E\{(\boldsymbol{\omega}^T\mathbf{X} - \boldsymbol{\omega}^T\mathbf{m}_{\mathbf{X}})(\mathbf{X}^T\boldsymbol{\omega} - \mathbf{m}_{\mathbf{X}}^T\boldsymbol{\omega})\} = \boldsymbol{\omega}^T\mathbf{K}_{\mathbf{X}}\boldsymbol{\omega}.$$

Hence, we have

$$\Phi_Y(1) = E\{e^{iY}\} = E\{e^{i\mathbf{X}^T\boldsymbol{\omega}}\} = e^{i\beta - \alpha^2/2} = \exp\{i\,\mathbf{m}_{\mathbf{X}}^T\boldsymbol{\omega} - \frac{1}{2}\boldsymbol{\omega}^T\mathbf{K}_{\mathbf{X}}\boldsymbol{\omega}\},$$

which is the desired result (2.38).

b) Performing the N-dimensional Fourier transform of the joint characteristic function (2.38) yields

$$\frac{1}{(2\pi)^n}\int_{-\infty}^{\infty}\Phi_{\mathbf{X}}(\boldsymbol{\omega})\exp\{-i\,\mathbf{x}^T\boldsymbol{\omega}\}d\boldsymbol{\omega}$$

$$=\frac{1}{(2\pi)^n}\int_{-\infty}^{\infty}\exp\{-\frac{1}{2}\boldsymbol{\omega}^T\mathbf{K}_{\mathbf{X}}\boldsymbol{\omega}\}\exp\{-i\,(\mathbf{x}-\mathbf{m}_{\mathbf{X}})^T\boldsymbol{\omega}\}d\boldsymbol{\omega}.$$

Using the change of variables $\boldsymbol{\omega}=\mathbf{K}_{\mathbf{X}}^{-1/2}\boldsymbol{\mu}$ and $d\boldsymbol{\omega}=|\mathbf{K}_{\mathbf{X}}|^{-1/2}d\boldsymbol{\mu}$ in the above equation yields

$$\frac{1}{(2\pi)^n\,|\mathbf{K}_{\mathbf{X}}|^{1/2}}\int_{-\infty}^{\infty}\exp\{-\frac{1}{2}\boldsymbol{\mu}^T\boldsymbol{\mu}\}\exp\{-i\,(\mathbf{x}-\mathbf{m}_{\mathbf{x}})^T\mathbf{K}_{\mathbf{X}}^{-1/2}\boldsymbol{\mu}\}d\boldsymbol{\mu}$$

$$=\frac{1}{(2\pi)^n\,|\mathbf{K}_{\mathbf{X}}|^{1/2}}\prod_{j=1}^{n}\exp\{-a_j^2/2\}\int_{-\infty}^{\infty}\exp\{-\frac{1}{2}(\mu_j+ia_j)^2\}d\mu_j,$$

where

$$\mathbf{a}=[\,a_1\quad a_2\quad\cdots\quad a_n\,]=(\mathbf{x}-\mathbf{m}_{\mathbf{X}})^T\mathbf{K}_{\mathbf{X}}^{-1/2}.$$

Since $\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-\nu^2/2}d\nu=1$, then the above equation becomes

$$\frac{1}{(2\pi)^n\,|\mathbf{K}_{\mathbf{X}}|^{1/2}}\prod_{j=1}^{n}(2\pi)^{1/2}\exp\{-a_j^2/2\}=\frac{1}{(2\pi)^{n/2}|\mathbf{K}_{\mathbf{X}}|^{1/2}}(2\pi)^{n/2}\exp\{-\frac{1}{2}\mathbf{a}^T\mathbf{a}\}$$

$$=\frac{1}{(2\pi)^{n/2}|\mathbf{K}_{\mathbf{X}}|^{1/2}}\exp\{-\frac{1}{2}(\mathbf{x}-\mathbf{m}_{\mathbf{X}})^T\mathbf{K}_{\mathbf{X}}^{-1}(\mathbf{x}-\mathbf{m}_{\mathbf{X}})\},$$

which is the joint probability density function for an n-tuple of Gaussian random variables **X**.

2.26 Since the conditional mean is defined by

$$E\{X\,|Y\}\triangleq\int_{-\infty}^{\infty}xf_{X|Y}(x\,|y)dx,$$

then, by using the definition of conditional probability, we obtain

$$E\{E\{X\,|Y\}\}=\int_{-\infty}^{\infty}\Big[\int_{-\infty}^{\infty}xf_{X|Y}(x\,|y)dx\,\Big]f_Y(y)dy=\iint_{-\infty}^{\infty}xf_{XY}(x,y)dxdy$$

$$=\int_{-\infty}^{\infty}xf_X(x)dx=E\{X\},$$

which verifies (2.45).

2.27 *a*) The mean squared error is given by

$$E\{(X-\hat{X})^2\} = E\{X^2\} + a^2E\{Y^2\} + b^2 + 2abE\{y\} - 2aE\{XY\} - 2bE\{X\}.$$

Solving the above equation for a and b that satisfy the necessary and sufficient conditions

$$\frac{\partial}{\partial a}E\{(X-\hat{X})^2\} = 0 \quad \text{and} \quad \frac{\partial}{\partial b}E\{(X-\hat{X})^2\} = 0$$

for minimization yields

$$a = \frac{\sigma_X}{\sigma_Y}\rho_{XY} \qquad \text{and} \qquad b = m_X - \frac{\sigma_X}{\sigma_Y}\rho_{XY}m_Y.$$

Therefore, the minimum-mean-squared-error estimate is given by

$$\hat{X} = m_X + \frac{\sigma_X}{\sigma_Y}\rho_{XY}(Y-m_Y).$$

b) From exercise 1.12*d*, we have the conditional mean

$$E\{X|Y\} = m_X + \frac{\sigma_X}{\sigma_Y}\rho_{XY}(Y-m_Y)$$

for jointly Gaussian X and Y, and this is the same as $\hat{X}$ from part *a*.

c) It follows from the result of part *a* that the minimum value of the mean-squared-error is given by

$$MSE_{\min} = E\{(X-\hat{X})^2\} = \sigma_X^2(1-\rho_{XY}^2).$$

Thus, the normalized minimum mean-squared-error, $MSE_{\min}/\sigma_X^2$, is small if and only if $\rho_{XY} \simeq 1$.

2.28 Since the conditional distribution for $X|X>a$ is given by

$$F_{X|X>a}(x) = P(X<x|X>a) = \frac{P(X<x \text{ and } X>a)}{P(X>a)}$$

$$= \frac{P(a<X<x)}{P(X>a)} = \frac{x-a}{1-a}, \qquad a<x<1,$$

then we have the conditional density

$$f_{X|X>a}(x) = \frac{1}{1-a}, \qquad a<x<1.$$

Hence, the conditional mean is given by

$$E\{X|X>a\} = \int_{-\infty}^{\infty} xf_{X|X>a}(x)dx = \frac{1+a}{2}$$

and the conditional variance is given by

$$E\{(X - E\{X|X>a\})^2 \mid X>a\} = E\{X^2|X>a\} - E^2\{X|X>a\}$$

$$= \frac{1+a+a^2}{3} - \frac{(1+a)^2}{4} = \frac{(1-a)^2}{12}.$$

As $a \to 1$, *Mean* $\to 1$ and *Variance* $\to 0$ since $f_{X|X>a}(x) \to \delta(x-1)$.

2.29 Since $\iint_{-\infty}^{\infty} f_{XY}(x,y)dxdy = 1$, then for the given joint density we have $c = 6$. Also, we have

$$f_{XY}(x,y) = f_X(x)f_Y(y), \qquad f_X(x) = 2e^{-2x}, \quad x \geq 0$$

and

$$f_Y(y) = 3e^{-3y}, \quad y \geq 0.$$

a)

$$P(X>1/2 \text{ and } Y>1/3) = P(X>1/2)P(Y>1/3)$$

$$= \int_{1/2}^{\infty} 2e^{-2x}dx \int_{1/3}^{\infty} 3e^{-3y}dy = \frac{1}{e^2}.$$

b)

$$P(X>1/2|Y>1/3) = P(X>1/2) = \frac{1}{e}.$$

c)

$$E\{XY\} = \iint_{-\infty}^{\infty} xyf_{XY}(x,y)dxdy = m_X m_Y = \frac{1}{2} \times \frac{1}{3} = \frac{1}{6}.$$

d) Since

$$F_{Y|Y>1/3}(y) = P(Y<y|Y>1/3) = \frac{P(1/3<Y<y)}{P(Y>1/3)}$$

$$= \frac{\int_{1/3}^{y} f_Y(u)du}{\int_{1/3}^{\infty} f_Y(u)du}, \qquad y > 1/3,$$

then

$$f_{Y|Y>1/3}(y) = \frac{f_Y(y)}{\int\limits_{1/3}^{\infty} f_Y(u)du} = 3e^{-3y+1}, \qquad y > 1/3.$$

Hence,

$$E\{XY|Y > 1/3\} = E\{X\}E\{Y|Y > 1/3\}$$

$$= \int\limits_{-\infty}^{\infty} xf_X(x)dx \int\limits_{-\infty}^{\infty} yf_{Y|Y>1/3}(y)dy = \frac{1}{3}.$$

e)

$$E\{XY|Y = 1/3\} = \frac{1}{3}E\{X\} = \frac{1}{6}.$$

2.30 To verify (2.47), we start with the definition

$$E\{E\{X|Y\}|Z\} = \iint\limits_{-\infty}^{\infty} xf_{X|Y}(x|y)dxf_{Y|Z}(y|z)dy.$$

But, since $Z = g(Y)$, then

$$f_{X|Y}(x|y) = f_{X|Y,Z}(x|y, z).$$

Also, we have

$$\int\limits_{-\infty}^{\infty} f_{X|Y,Z}(x|y, z)f_{Y|Z}(y|z)dy = f_{X|Z}(x|z).$$

Therefore,

$$E\{E\{X|Y\}|Z\} = \int\limits_{-\infty}^{\infty} xf_{X|Z}(x|z)dx = E\{X|Z\},$$

which is (2.47). To verify (2.48), we observe that since $Z = g(Y)$, then

$$f_{Z|Y}(z|y) = \delta(z - g(y))$$

and, therefore,

$$E\{E\{X|Z\}|Y\} = \int\limits_{-\infty}^{\infty} E\{X|z\}f_{Z|Y}(z|y)dz = E\{X|g(Y)\} = E\{X|Z\},$$

which is (2.48).

2.31 The mean of the random sample mean is

$$E\{\hat{M}_Y\} = \frac{1}{n}[E\{Y_1\} + E\{Y_2\} + \cdots + E\{Y_n\}] = \frac{1}{n}\sum_{i=1}^{n} m_Y = m_Y$$

and the variance is

$$E\{(\hat{M}_Y - m_Y)^2\} = E\{[\frac{1}{n}\sum_{i=1}^{n}(Y_i - m_Y)]^2\} = \frac{1}{n^2}\sum_{i=1}^{n} E\{(Y_i - m_Y)^2\} = \frac{\sigma_Y^2}{n},$$

since the crosscorrelations are zero. Consequently, we have convergence in Mean Square of the sample mean to the probabilistic mean:

$$\lim_{n\to\infty} E\{(\hat{M}_Y - m_Y)^2\} = \lim_{n\to\infty} \frac{\sigma_Y^2}{n} = 0;$$

that is,

$$\underset{n\to\infty}{\text{l.i.m.}}\ \hat{M}_Y = m_Y .$$

2.32 Since $E\{X_i\} = P(A)$, $i = 1, 2, \ldots,$ and $E\{[X_i - P(A)]^2\} = \sigma^2$, then we have from exercise 2.31

$$E\left\{|\frac{K_n}{n} - P(A)|^2\right\} = \frac{\sigma^2}{n}.$$

From the Bienaymé-Chebychev inequality, we have

$$Prob\left\{|\frac{K_n}{n} - P(A)| > \varepsilon\right\} \le \frac{1}{\varepsilon^2} E\left\{|\frac{K_n}{n} - P(A)|^2\right\} = \frac{1}{\varepsilon^2}\frac{\sigma^2}{n}.$$

Therefore,

$$\lim_{n\to\infty} Prob\left\{|\frac{K_n}{n} - P(A)| > \varepsilon\right\} \le \lim_{n\to\infty} \frac{1}{\varepsilon^2}\frac{\sigma^2}{n} = 0$$

or, equivalently,

$$\lim_{n\to\infty} Prob\left\{|\frac{K_n}{n} - P(A)| < \varepsilon\right\} = 1,$$

which proves the weak law of large numbers.

2.33 We are given

$$X_n = \begin{cases} n, & Prob = \alpha / n^2 \\ 0, & Prob = 1 - \alpha / n^2. \end{cases}$$

Therefore,

$$\lim_{n\to\infty} X_n = \begin{cases} \text{does not exist}, & Prob = 0 \\ 0, & Prob = 1. \end{cases}$$

Thus, X_n converges to 0 with *Probability* $= 1$. On the other hand,

$$\lim_{n \to \infty} E\{(X_n - 0)^2\} = \lim_{n \to \infty} n^2 P(s = n) = \alpha \neq 0.$$

Hence, X_n does not converge in mean square.

Chapter 3

Introduction to Random Processes

3.1 To determine the empirical autocorrelation of the given square wave, we proceed as follows. Let

$$p(t) \triangleq \begin{cases} -1, & -T/2 \le t < 0 \\ 1, & 0 \le t < T/2. \end{cases}$$

Then the square wave $X(t)$ can be expressed as

$$X(t) = \sum_{n=-\infty}^{\infty} p(t-nT)$$

The empirical autocorrelation is given by

$$\hat{R}_X(\tau) = \langle X(t+\tau)X(t)\rangle = \langle \sum_n \sum_m p(t+\tau-nT)p(t-mT)\rangle.$$

Let $m = n + q$, then

$$\hat{R}_X(\tau) = \sum_q \langle \sum_n p(t-nT+\tau)p(t-nT-qT)\rangle.$$

But $\sum_n p(t-nT+\tau)p(t-nT-qT)$ is periodic in t with period T; therefore we can average over only one period rather than over all time:

$$\hat{R}_X(\tau) = \sum_q \frac{1}{T} \int_{-T/2}^{T/2} \sum_n p(t-nT+\tau)p(t-nT-qT)dt.$$

Let $\sigma = t - nT$. Then

$$\hat{R}_X(\tau) = \sum_q \frac{1}{T} \sum_n \int_{(n-1/2)T}^{(n+1/2)T} p(\sigma+\tau)p(\sigma-qT)d\sigma = \sum_q \frac{1}{T} \int_{-\infty}^{\infty} p(\sigma+\tau)p(\sigma-qT)d\sigma.$$

Let $\sigma - qT = t$, then

$$\hat{R}_X(\tau) = \frac{1}{T} \sum_q \int_{-\infty}^{\infty} p(t+\tau+qT)p(t)dt = \frac{1}{T} \sum_q r_p(\tau+qT),$$

where

$$r_p(\tau) \triangleq \int_{-\infty}^{\infty} p(t+\tau)p(t)dt.$$

This result can be used to show graphically that $\hat{R}_X(\tau)$ is a periodic symmetrical triangle wave with peak-to-peak amplitude of $2T$ and with zero average value.

3.2 For the given sine wave, we have the empirical autocorrelation

$$\hat{R}_X(\tau) = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} X(t+\tau)X(t)dt$$

$$= \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} \sin(2\pi f_0[t+\tau]+\theta_0)\sin(2\pi f_0 t+\theta_0)dt$$

$$= \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} [\frac{1}{2}\cos(2\pi f_0\tau) - \frac{1}{2}\cos(4\pi f_0 t + 2\pi f_0 \tau + 2\theta_0)]dt$$

$$= \frac{1}{2}\cos(2\pi f_0 \tau).$$

Fourier transforming $\hat{R}_X(\tau)$ yields the empirical power spectral density

$$\hat{S}_X(f) = \int_{-\infty}^{\infty} \hat{R}_X(\tau)e^{-i2\pi f\tau}d\tau = \int_{-\infty}^{\infty} [\frac{1}{4}e^{i2\pi(f_0-f)\tau} + \frac{1}{4}e^{-i2\pi(f_0+f)\tau}]d\tau$$

$$= \frac{1}{4}[\delta(f-f_0) + \delta(f+f_0)].$$

3.3 It follows from (3.5) and the convolution theorem for Fourier transforms that

$$\tilde{R}_X(f)_T = \frac{1}{T}\tilde{X}_T(f)\tilde{X}_T(-f),$$

because $\tilde{X}_T(-f)$ is the Fourier transform of $X_T(-\tau)$. But, since $X_T(\tau)$ is a real function, then $\tilde{X}_T(-f) = \tilde{X}_T(f)^*$. Therefore,

$$\tilde{R}_X(f)_T = \frac{1}{T}|\tilde{X}_T(f)|^2,$$

which is the desired result (3.4).

3.4 To derive the input-output relation (3.12) for autocorrelations, we follow the hint to obtain (from (3.11))

$$Y(t) = \int_{\infty}^{-\infty} h(u_2)X(t-u_2)(-du_2) = \int_{-\infty}^{\infty} h(u_2)X(t-u_2)du_2$$

and

$$Y(t+\tau) = \int_{-\infty}^{\infty} h(u_1)X(t+\tau-u_1)du_1.$$

Substituting $Y(t+\tau)$ and $Y(t)$ into (3.2) yields

$$\hat{R}_Y(\tau) = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} Y(t+\tau)Y(t)dt$$

$$= \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} \iint_{-\infty}^{\infty} h(u_1)h(u_2)X(t+\tau-u_1)X(t-u_2)du_1du_2\, dt$$

$$= \iint_{-\infty}^{\infty} h(u_1)h(u_2) \left[\lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} X(t+\tau-u_1)X(t-u_2)dt \right] du_1du_2.$$

Letting $t - u_2 = v$, the bracketed factor becomes

$$\lim_{T\to\infty} \frac{1}{T} \int_{-T/2-u_2}^{T/2-u_2} X(v+\tau+u_2-u_1)X(v)dv = \hat{R}_X(\tau+u_2-u_1).$$

Therefore,

$$\hat{R}_Y(\tau) = \iint_{-\infty}^{\infty} h(u_1)h(u_2)\hat{R}_X(\tau+u_2-u_1)du_1du_2.$$

Letting $s = u_1 - u_2$ yields

$$\hat{R}_Y(\tau) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(s+u_2)h(u_2)du_2\, \hat{R}_X(\tau-s)ds = \int_{-\infty}^{\infty} r_h(s)\hat{R}_X(\tau-s)ds$$

$$= \hat{R}_X(\tau) \otimes r_h(\tau),$$

where $r_h(\tau)$ is defined in (3.13). This is the desired result (3.12).

3.5 Inverse Fourier transforming (3.9) with X replaced by Y yields

$$\hat{R}_Y(\tau) = \int_{-\infty}^{\infty} \hat{S}_Y(f)e^{i2\pi f\tau}df.$$

Evaluating this equation at $\tau = 0$ and using (3.17) and (3.18) yields

$$\langle P \rangle = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} Y^2(t)dt = \hat{R}_Y(0) = \int_{-\infty}^{\infty} \hat{S}_Y(\nu)d\nu,$$

as desired. Using (3.14) with f replaced by ν then yields

$$\langle P \rangle = \int_{f-\Delta/2}^{f+\Delta/2} \hat{S}_X(\nu)d\nu + \int_{-f-\Delta/2}^{-f+\Delta/2} \hat{S}_X(\nu)d\nu.$$

Thus, with $\Delta \to 0$, we have

$$\frac{1}{\Delta}\langle P \rangle \to \hat{S}_X(f) + \hat{S}_X(-f).$$

But, since $\hat{R}_X(\tau)$ is real and even, then $\hat{S}_X(f)$ is real and even and, therefore,

$$\frac{1}{2\Delta}\langle P\rangle \;\rightarrow\; \hat{S}_X(f),$$

which is the desired result (3.10).

3.6 To verify formula (3.9), we proceed as follows. We consider the time-variant finite segment

$$X_T(t+u) \triangleq \begin{cases} X(t+u), & |t| \le T/2 \\ 0, & |t| > T/2. \end{cases}$$

The time-variant correlogram, which is a generalization of the correlogram (3.5), is then defined by

$$\begin{aligned} R_X(u,\tau)_T &\triangleq \frac{1}{T}\int_{-\infty}^{\infty} X_T(t+u+|\tau|)X_T(t+u)\,dt \\ &= \frac{1}{T}\int_{-T/2}^{T/2-|\tau|} X(t+u+|\tau|)X(t+u)\,dt. \end{aligned}$$

Using the generalization of (3.4),

$$\frac{1}{T}|\tilde{X}_T(u,f)|^2 = \int_{-\infty}^{\infty} R_X(u,\tau)_T e^{-i2\pi f\tau}\,d\tau,$$

yields the time-averaged periodogram,

$$\begin{aligned} \lim_{Z\to\infty}\frac{1}{Z}\int_{-Z/2}^{Z/2}\frac{1}{T}|\tilde{X}_T(u,f)|^2du &= \int_{-\infty}^{\infty}\Big[\lim_{Z\to\infty}\frac{1}{Z}\int_{-Z/2}^{Z/2} R_X(u,\tau)_T\,du\Big]\,e^{-i2\pi f\tau}d\tau \\ &= \int_{-\infty}^{\infty}\frac{1}{T}\int_{-T/2}^{T/2-|\tau|}\lim_{Z\to\infty}\frac{1}{Z}\int_{-Z/2}^{Z/2} X(t+u+|\tau|)X(t+u)\,du\;dt\,e^{-i2\pi f\tau}d\tau \\ &= \int_{-\infty}^{\infty}\frac{1}{T}\int_{-T/2}^{T/2-|\tau|}\hat{R}_X(\tau)\,dt\;e^{-i2\pi f\tau}d\tau = \int_{-\infty}^{\infty}\Big(1-\frac{|\tau|}{T}\Big)\hat{R}_X(\tau)e^{-i2\pi f\tau}d\tau. \end{aligned}$$

Finally, taking the limit as $T \rightarrow \infty$ yields

$$\begin{aligned} \hat{S}_X(f) \triangleq \lim_{T\to\infty}\lim_{Z\to\infty}\frac{1}{Z}\int_{-Z/2}^{Z/2}\frac{1}{T}|\tilde{X}_T(u,f)|^2du &= \lim_{T\to\infty}\int_{-\infty}^{\infty}\Big(1-\frac{|\tau|}{T}\Big)\hat{R}_X(\tau)e^{-i2\pi f\tau}d\tau \\ &= \int_{-\infty}^{\infty}\hat{R}_X(\tau)e^{-i2\pi f\tau}d\tau, \end{aligned}$$

which is the desired result (3.9).

3.7 *a*) Let $Y(t)$ be the waveform at the output of the specified filter. Then the power spectral density of the output waveform is given by

$$\hat{S}_Y(f) = \hat{S}_X(f)|H(f)|^2,$$

where

$$H(f) = \frac{1}{b + i2\pi f}.$$

Therefore,

$$\hat{S}_Y(f) = \frac{N_0}{(2\pi f)^2 + b^2}.$$

b) The empirical autocorrelation for the output $Y(t)$ is given by

$$\hat{R}_Y(\tau) = \int_{-\infty}^{\infty} \hat{S}_Y(f) e^{i2\pi f\tau} df = \frac{N_0}{2b} e^{-b|\tau|}.$$

The average power dissipated in a one-ohm resistor by this output voltage is $\hat{R}_Y(0) = N_0/2b$.

3.8 From (3.13), we have (for the given rectangular impulse response) the finite autocorrelation

$$r_h(\tau) = \int_{-\infty}^{\infty} h(\tau+v)h(v)dv = \begin{cases} T+\tau, & -T \le \tau < 0 \\ T-\tau, & 0 \le \tau \le T \end{cases}$$

$$= \begin{cases} T-|\tau|, & |\tau| \le T \\ 0, & |\tau| > T, \end{cases}$$

which is a symmetrical triangle function. From (3.12), we obtain the output autocorrelation:

$$\hat{R}_Y(\tau) = \hat{R}_X(\tau) \otimes r_h(\tau) = \delta(\tau) \otimes r_h(\tau) = r_h(\tau).$$

3.9 *a*) The average value of $\hat{R}_X(\tau)$ is given by

$$\langle \hat{R}_X(\tau) \rangle = \lim_{Z\to\infty} \frac{1}{Z} \int_{-Z/2}^{Z/2} \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} X(t+\tau)X(t)dt d\tau.$$

Interchanging the order of the two averaging operations yields

$$\langle \hat{R}_X(\tau) \rangle = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} X(t) \lim_{Z\to\infty} \frac{1}{Z} \int_{-Z/2}^{Z/2} X(t+\tau)d\tau \, dt$$

$$= \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} X(t)\hat{m}_X \, dt = \hat{m}_X^2,$$

as desired.

b) It follows from part *a* that the PSD $\hat{S}_X(f)$ contains the additive term $\hat{m}_X^2\delta(f)$.

3.10 From (3.31), we have the SNR ratio

$$\frac{SNR_{out}}{SNR_{in}} \simeq 3\left[\frac{B_X}{B_\Psi}\right]^3 = 81$$

for the noisy FM signal. Therefore, $B_X \simeq 3B_\Psi$. If $B_X = 6B_\Psi$, then SNR_{out}/SNR_{in} is $8 \times 81 = 648$.

3.11 The signal to be detected is given by

$$S(t) = \cos(2\pi \times 100t)[u(t) - u(t-1)],$$

where $u(t)$ is the unit-step function. The impulse-response function of the filter can be obtained from (3.33),

$$h(t) = \int_{-\infty}^{\infty} H(f)e^{i2\pi ft} \, df = \int_{-\infty}^{\infty} \frac{S^*(f)e^{-i2\pi f t_0}}{\hat{S}_N(f)} e^{i2\pi ft} \, df$$

$$= \int_{-\infty}^{\infty} [S(f)e^{i2\pi f(t_0-t)}]^* \, df = s(t_0 - t)$$

$$= \cos(2\pi \times 100[1-t])[u(1-t) - u(-t)], \quad \text{for} \quad t_0 = 1.$$

The maximized SNR is, from (3.34),

$$SNR_{\max} = \int_{-\infty}^{\infty} \frac{|S(f)|^2}{\hat{S}_N(f)} df = \int_{-\infty}^{\infty} |s(t)|^2 dt = \int_0^1 \cos^2(2\pi \times 100t) dt = \frac{1}{2}.$$

If the sine wave amplitude were doubled, the $SNR_{\max}$ would be quadrupled. If the frequency were doubled, the $SNR_{\max}$ would remain the same.

3.12 Since $\hat{S}_N(f) = N_0$ and

$$\hat{S}_S(f) = \int_{-\infty}^{\infty} \hat{R}_S(\tau)e^{-i2\pi f\tau} d\tau = \frac{S_0}{(2\pi f \tau_0)^2 + 1},$$

then it follows from (3.40) that the transfer function for the optimum filter for this signal in white noise is given by

$$H(f) = \frac{S_0/(N_0+S_0)}{1 + (2\pi f\, \tau_0)^2 N_0/(N_0+S_0)},$$

which is a low pass filter with cutoff frequency $f_0 = (1/\tau_0)[1+S_0/N_0]^{1/2}$. When $S_0/N_0 \to \infty$, $f_0 \to \infty$ and $H(f) \to 1$ for all f.

The minimized MSE is, from (3.41), given by

$$MSE_{\min} = \int_{-\infty}^{\infty} \frac{\hat{S}_S(f)\hat{S}_N(f)}{\hat{S}_S(f) + \hat{S}_N(f)} df = \int_{-\infty}^{\infty} \frac{S_0 N_0/(S_0+N_0)}{1 + (2\pi f\, \tau_0)^2 N_0/(S_0+N_0)} df$$

$$= \frac{S_0}{\tau_0}\left[\frac{N_0}{S_0 + N_0}\right]^{1/2},$$

which approaches zero as S_0/N_0 grows without bound.

3.13 It follows from (3.44) that (using $\hat{R}_X(-\tau) = \hat{R}_X(\tau)$) the two coefficients $\{h_0, h_1\}$ of the second order predictor are given by the solution to the following set of two simultaneous equations:

$$\hat{R}_X(0)h_0 + \hat{R}_X(\tau_0/2)h_1 = \hat{R}_X(\tau_0/2) \quad \text{and} \quad \hat{R}_X(\tau_0/2)h_0 + \hat{R}_X(0)h_1 = \hat{R}_X(\tau_0).$$

The solution to these equations is

$$h_0 = \hat{\rho}_X(\tau_0/2)\frac{1 - \hat{\rho}_X(\tau_0)}{1 - \hat{\rho}_X^2(\tau_0/2)} \quad \text{and} \quad h_1 = \frac{\hat{\rho}_X(\tau_0) - \hat{\rho}_X^2(\tau_0/2)}{1 - \hat{\rho}_X^2(\tau_0/2)},$$

where $\hat{\rho}_X(\tau) \triangleq \hat{R}_X(\tau)/\hat{R}_X(0)$. It follows from (3.45) that the minimized mean squared prediction error is given by

$$MSE_{\min} = \hat{R}_X(0)[1 - h_0\hat{\rho}_X(\tau_0/2) - h_1\hat{\rho}_X(\tau_0)].$$

Using $\hat{R}_X(\tau) = \sigma^2\exp\{-|\tau|/\tau_0\}$, we obtain $h_0 = 1/\sqrt{e}$, $h_1 = 0$, $\hat{\rho}_X(\tau_0/2) = 1/\sqrt{e}$, and $\hat{\rho}_X(\tau_0) = 1/e$. Therefore,

$$MSE_{\min} = \sigma^2[1 - 1/e].$$

Although, in general, the $MSE_{\min}$ would decrease as the order n of the predictor increases, this is not so for the process in this exercise. In fact, the smallest possible $MSE_{\min}$ is obtainable with only $n = 1$ for this process.

Chapter 4

Mean and Autocorrelation

4.1 *Bernoulli Process*: Given

$$P\{X_n = 1\} = p \quad \text{and} \quad P\{X_n = 0\} = 1 - p,$$

we obtain

$$m_X(n) = E\{X_n\} = P\{X_n = 1\}1 + P\{X_n = 0\}0 = p$$

and

$$R_X(n_1, n_2) = E\{X_{n_1}X_{n_2}\} = \begin{cases} 1^2p + 0^2(1-p) = p, & n_1 = n_2 \\ E\{X_{n_1}\}E\{X_{n_2}\} = p^2, & n_1 \neq n_2, \end{cases}$$

and also

$$K_X(n_1, n_2) = R_X(n_1, n_2) - m_X(n_1)m_X(n_2) = \begin{cases} p(1-p), & n_1 = n_2 \\ 0, & n_1 \neq n_2. \end{cases}$$

Binomial Counting Process: From (4.14), we obtain

$$m_Y(n) = E\{Y_n\} = E\{\sum_{i=1}^{n} X_i\} = \sum_{i=1}^{n} E\{X_i\} = \sum_{i=1}^{n} p = np.$$

Using this result yields

$$\begin{aligned} K_Y(n_1, n_2) &= \sum_{i=1}^{n_1}\sum_{j=1}^{n_2} E\{X_iX_j\} - (n_1p)(n_2p) = \sum_{i=1}^{n_1}\sum_{j=1}^{n_2} K_X(i, j) \\ &= \sum_{i=j=1}^{n_1} K_X(i, i) + \sum_{i=1}^{n_1}\sum_{\substack{j=1 \\ j\neq i}}^{n_2} K_X(i, j), \qquad n_1 \leq n_2 \\ &= n_1p(1-p) + 0, \qquad n_1 \leq n_2. \end{aligned}$$

Similarly,

$$K_Y(n_1, n_2) = n_2p(1-p), \qquad n_2 \leq n_1.$$

Thus,

$$K_Y(n_1, n_2) = p(1-p)\min\{n_1, n_2\}.$$

Random-Walk Process: From (4.20), we have $Z_i = 2(X_i - 1/2)$ and, therefore,

$$W_n = \sum_{i=1}^{n} Z_i = 2Y_n - n.$$

Thus,

$$m_W(n) = E\{W_n\} = 2E\{Y_n\} - n = n(2p-1)$$

and

$$\begin{aligned} K_W(n_1, n_2) &= E\{[W_{n_1} - m_W(n_1)][W_{n_2} - m_W(n_2)]\} \\ &= E\{[2Y_{n_1} - n_1 - n_1(2p-1)][2Y_{n_2} - n_2 - n_2(2p-1)]\} \\ &= 4E\{(Y_{n_1} - n_1 p)(Y_{n_2} - n_2 p)\} = 4K_Y(n_1, n_2) \\ &= 4p(1-p)\min\{n_1, n_2\}. \end{aligned}$$

4.2 Using the notation $E\{Z_i\} = m_Z$ and $Var\{Z_i\} = \sigma_Z^2$ for the independent random step sizes $\{Z_i\}$, we obtain the following mean and variance for the generalized random walk process:

$$m_W(n) = E\{W_n\} = \sum_{i=1}^{n} E\{Z_i\} = nm_Z$$

$$K_W(n_1, n_2) = E\{[W_{n_1} - m_W(n_1)][W_{n_2} - m_W(n_2)]\}$$

$$= E\{ \sum_{i=1}^{n_1}\sum_{j=1}^{n_2} (Z_i - m_Z)(Z_j - m_Z)\}$$

$$= E\{ \sum_{i,j=1}^{n_1} (Z_i - m_Z)(Z_j - m_Z) + \sum_{i=1}^{n_1} (Z_i - m_Z) \sum_{\substack{j=1 \\ j\neq i}}^{n_2} (Z_j - m_Z)\}, \quad n_1 \le n_2$$

$$= \sum_{i=1}^{n_1} \sigma_Z^2 = n_1\sigma_Z^2, \quad n_1 \le n_2.$$

Similarly,

$$K_W(n_1, n_2) = n_2\sigma_Z^2, \quad n_2 \le n_1.$$

Thus,

$$K_W(n_1, n_2) = \sigma_Z^2 \min\{n_1, n_2\}.$$

These results are essentially the same as those in (4.22) and (4.23) for the fixed-step-size random walk process, where m_Z and σ_Z^2 take on specific values.

4.3 Using (4.19), we can represent the process U_n by

$$U_n = \sum_{i=n-m}^{n} Z_i = \sum_{i=-\infty}^{\infty} h(n-i)Z_i,$$

where

$$h(i) \triangleq \begin{cases} 1, & 0 \le i \le m \\ 0, & otherwise. \end{cases}$$

Therefore, we obtain the mean

$$M_U(n) = \sum_{i=-\infty}^{\infty} h(n-i)m_Z = m_Z \sum_{j=-\infty}^{\infty} h(j) = (2p-1)(m+1).$$

Also, we obtain the covariance

$$K_U(n_1, n_2) = E\{[U_{n_1} - m_U(n_1)][U_{n_2} - m_U(n_2)]\}$$
$$= \sum_{i=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} h(i-n_1)h(j-n_2)K_Z(i, j).$$

But

$$K_Z(i, j) = \begin{cases} p(1)^2 + (1-p)(-1)^2 - [p(1)+(1-p)(-1)]^2 = 4p(1-p), & i = j \\ 0, & i \ne j \end{cases}$$

$$= 4p(1-p)\delta_{i-j},$$

where

$$\delta_k \triangleq \begin{cases} 1, & k = 0 \\ 0, & k \ne 0. \end{cases}$$

Consequently,

$$K_U(n_1, n_2) = 4p(1-p) \sum_{i=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} h(i-n_1)h(j-n_2)\delta_{i-j}$$
$$= 4p(1-p) \sum_{i=-\infty}^{\infty} h(i-n_1)h(i-n_2)$$
$$= 4p(1-p) \sum_{j=-\infty}^{\infty} h(j+n_1-n_2)h(j) = 4p(1-p)r_h(n_1-n_2),$$

where

$$r_h(k) \triangleq \sum_{j=-\infty}^{\infty} h(j+k)h(j) = \begin{cases} m+1-|k|, & |k| \le m \\ 0, & otherwise. \end{cases}$$

(See also the solution method used for exercise 4.4.)

4.4 The mean of the moving average process Y_n is given by

$$m_Y = E\{Y_n\} = E\{\sum_{i=0}^{N-1} X_{n-i}\} = \sum_{i=0}^{N-1} E\{X_{n-i}\} = \sum_{i=0}^{N-1} m_X = Nm_X.$$

For $0 \le m < N$, the autocorrelation of Y_n is given by

$$R_Y(n+m, n) = E\{Y_{n+m}Y_n\} = E\{\sum_{i=0}^{N-1} X_{n+m-i} \sum_{j=0}^{N-1} X_{n-j}\}$$

$$= E\{\sum_{i=n+m-N+1}^{n+m} X_i \sum_{j=n-N+1}^{n} X_j\} = \sum_{i=n+m-N+1}^{n+m} \sum_{j=n-N+1}^{n} E\{X_i X_j\}$$

$$= \sum_{i=j=n+m-N+1}^{n} E\{X_i^2\} + \sum_{i=n+m-N+1}^{n+m} \sum_{\substack{j=n-N+1 \\ i\ne j}}^{n} E\{X_i\}E\{X_j\}, \quad m \ge 0$$

$$= (N-m)(\sigma_X^2 + m_X^2) + (N^2 - N + m)m_X^2 = (N-m)\sigma_X^2 + N^2 m_X^2.$$

Similarly,

$$R_Y(n+m, n) = (N+m)\sigma_X^2 + N^2 m_X^2, \quad m \le 0.$$

Thus,

$$R_Y(n+m, n) = \begin{cases} (N-|m|)\sigma_X^2 + (Nm_X)^2, & 0 \le |m| < N \\ (Nm_X)^2, & |m| \ge N. \end{cases}$$

4.5 For $r > 0$, $r^{|k|}$ either grows or decays monotonically, depending on whether $r > 1$ or $r < 1$. For $r < 0$, the magnitude $|r^{|k|}| = |r|^{|k|}$ behaves the same way; however, the sign of $r^{|k|}$ oscillates from positive for even k to negative for odd k. The only way the autocorrelation can oscillate is if the sample paths themselves exhibit oscillatory behavior. For example, if $Y_n = X_n \cos(\pi n + \Theta)$, where Θ is independent of $\{X_n\}$ and uniformly distributed over $[0, 2\pi)$, then

$$R_Y(k) = \frac{1}{2} R_X(k)\cos(\pi k) = \begin{cases} \frac{1}{2} r^{|k|}, & k \text{ even} \\ -\frac{1}{2} r^{|k|}, & k \text{ odd} \end{cases}$$

and

$$Y_n = \begin{cases} X_n \cos(\theta), & n \text{ even} \\ -X_n \cos(\theta), & n \text{ odd.} \end{cases}$$

4.6 Using the notation $E\{A^2\} = E\{B^2\} = \sigma^2$ and $\Phi_\Omega(\omega) = E\{e^{i\omega\Omega}\}$, we have for this sine wave process

$$R_X(t_1, t_2) = E\{(A\cos\Omega t_1 + B\sin\Omega t_1)(A\cos\Omega t_2 + B\sin\Omega t_2)\}$$
$$= E\{A^2\cos\Omega t_1\cos\Omega t_2 + B^2\sin\Omega t_1\sin\Omega t_2 + AB\cos\Omega t_1\sin\Omega t_2 + BA\cos\Omega t_2\sin\Omega t_1\}$$
$$= \sigma^2 E\{\cos\Omega t_1\cos\Omega t_2 + \sin\Omega t_1\sin\Omega t_2\}$$
$$= \sigma^2 E\{\cos[\Omega(t_1 - t_2)]\} = \sigma^2\mathrm{Re}\{\Phi(t_1 - t_2)\}.$$

Since $E\{X(t)\} = 0$, then $R_X(t_1, t_2) \equiv K_X(t_1, t_2)$.

4.7 The mean of the random periodic process $X(t)$ is given by

$$m_X(t) = E\{X(t)\} = E\{E\{X(t) \mid \Theta\}\}$$
$$= E\{\sum_{p=-P}^{P} E\{C_p\}\exp\{ip(\omega_0 t + \Theta)\}\} \quad \text{(since } \{C_p\} \text{ and } \Theta \text{ are indpendent)}$$
$$= \sum_{p=-P}^{P} E\{C_p\}e^{ip\omega_0 t}\frac{1}{2\pi}\int_0^{2\pi} e^{ip\theta}d\theta = \sum_{p=-P}^{P} E\{C_p\}e^{ip\omega_0 t} \times 0 = 0.$$

The autocorrelation of $X(t)$ is given by

$$R_X(t_1, t_2) = E\{X(t_1)X(t_2)\} = E\{E\{X(t_1)X(t_2) \mid \Theta\}\}$$
$$= E\{E\{\sum_{p,r=-P}^{P} C_p C_r \exp\{ip\omega_0 t_1 + ir\omega_0 t_2 + i(p+r)\Theta\} \mid \Theta\}\}$$
$$= E\{\sum_{p,r=-P}^{P} E\{C_p C_r\}\exp\{i[p\omega_0 t_1 + r\omega_0 t_2 + (p+r)\Theta]\}\}$$
$$= \sum_{p,r=-P}^{P} E\{C_p C_r\}\exp\{i(p\omega_0 t_1 + r\omega_0 t_2)\}\frac{1}{2\pi}\int_0^{2\pi}\exp\{i(p+r)\theta\}d\theta.$$

Since the integral equals zero except when $p = -r$, then we obtain (using $C_{-p} = C_p^*$)

$$R_X(t_1, t_2) = \sum_{p=-P}^{P} E\{|C_p|^2\}e^{ip\omega_0(t_1 - t_2)}.$$

4.8 Since $Y(t)$ and Φ are independent in Section 4.2.8, then we have, for the amplitude-modulated sine wave $Z(t)$, the mean

$$E\{Z(t)\} = E\{Y(t)\}E\{\sin(\omega_0 t + \Phi)\} = m_Y(t)E\{\sin(\omega_0 t + \Phi)\}$$

and the autocorrelation

$$R_Z(t_1, t_2) = E\{Z(t_1)Z(t_2)\} = E\{Y(t_1)Y(t_2)\}E\{\sin(\omega_0 t_1 + \Phi)\sin(\omega_0 t_2 + \Phi)\}$$
$$= R_Y(t_1, t_2)\frac{1}{2}E\{\cos(\omega_0[t_1 - t_2]) - \cos(\omega_0[t_1 + t_2] + 2\Phi)\}$$
$$= \frac{1}{2}R_Y(t_1, t_2)[\cos(\omega_0[t_1 - t_2]) - E\{\cos(\omega_0[t_1 + t_2] + 2\Phi)\}].$$

If Φ is uniformly distributed on the interval $[-\pi, \pi)$, then we obtain

$$E\{\sin(\omega_0 t + \Phi)\} = \int_{-\infty}^{\infty} \sin(\omega_0 t + \phi) f_\Phi(\phi) d\phi = \frac{1}{2\pi} \int_{-\pi}^{\pi} \sin(\omega_0 t + \phi) d\phi = 0$$

and

$$E\{\cos(\omega_0[t_1 - t_2] + 2\Phi)\} = \int_{-\infty}^{\infty} \cos(\omega_0[t_1 - t_2] + 2\phi) f_\Phi(\phi) d\phi$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \cos(\omega_0[t_1 - t_2] + 2\phi) d\phi = 0.$$

Therefore,

$$m_Z(t) = m_Y(t)0 = 0$$

and

$$R_Z(t_1, t_2) = \frac{1}{2} R_Y(t_1, t_2)[\cos(\omega_0[t_1 - t_2]) - 0] = \frac{1}{2} R_Y(t_1, t_2)\cos(\omega_0[t_1 - t_2]).$$

4.9 Since $X(t)$ and $Z(t)$ are statistically independent, we then have

$$R_Y = E\{Y(t_1)Y(t_2)\} = E\{X(t_1)Z(t_1)X(t_2)Z(t_2)\}$$

$$= E\{X(t_1)X(t_2)\}E\{Z(t_1)Z(t_2)\} = R_X R_Z.$$

4.10 For the sampled-and-held noise process, since $V(nT)$ and $V([n+k]T)$ are independent for $k \neq 0$ and have zero mean values, then we obtain

$$K_X(t_1, t_2) = E\{[X(t_1) - E\{X(t_1)\}][X(t_2) - E\{X(t_2)\}]\}$$

$$= E\{ \sum_{n=-\infty}^{\infty} [V(nT) - E\{V(nT)\}]h(t_1 - nT)$$

$$\times \sum_{m=-\infty}^{\infty} [V(mT) - E\{V(mT)\}]h(t_2 - mT) \}$$

$$= \sum_{n,m=-\infty}^{\infty} E\{V(nT)V(mT)\}h(t_1 - nT)h(t_2 - mT)$$

$$= \sum_{n=-\infty}^{\infty} \sigma_V^2 h(t_1 - nT)h(t_2 - nT).$$

Also, $E\{X(t)\} = 0$ and, therefore, $K_X(t_1, t_2) = R_X(t_1, t_2)$. From (2.45) and using the above result, we obtain $E\{Y(t)\} = 0$ and, therefore,

$$K_Y(t_1, t_2) = E\{Y(t_1)Y(t_2)\} = E\{E\{Y(t_1)Y(t_2) \mid \Theta\}\}$$

$$= E\{R_X(t_1-\Theta, t_2-\Theta)\} = \int_{-\infty}^{\infty} R_X(t_1-\theta, t_2-\theta) f_\Theta(\theta) d\theta$$

$$= \frac{1}{T}\sigma_V^2 \sum_{n=-\infty}^{\infty} \int_{-T/2}^{T/2} h(t_1-nT-\theta)h(t_2-nT-\theta)d\theta$$

$$= \frac{1}{T}\sigma_V^2 \sum_{n=-\infty}^{\infty} \int_{t_2-nT-T/2}^{t_2-nT+T/2} h(t_1-t_2+\phi)h(\phi)d\phi, \qquad \phi = t_2-nT-\theta$$

$$= \frac{1}{T}\sigma_V^2 \int_{-\infty}^{\infty} h(t_1-t_2+\phi)h(\phi)d\phi = \frac{\sigma_V^2}{T} r_h(t_1-t_2).$$

4.11 *a*) For the pulse-code-modulation process $X(t)$, we can show that the mean is zero and, therefore, the autocovariance equals the autocorrelation:

$$K_X(t_1, t_2) = E\{X(t_1)X(t_2)\} = E\{\sum_{n,m=-\infty}^{\infty} A_n A_m p(t_1-nT)p(t_2-mT)\}$$

$$= \sum_{n=-\infty}^{\infty} E\{A_n^2\} p(t_1-nT)p(t_2-nT)$$

$$+ \sum_{r\neq 0} \sum_{n=-\infty}^{\infty} E\{A_n\}E\{A_{n+r}\}p(t_1-nT)p(t_2-nT-rT)$$

$$= \sum_{n=-\infty}^{\infty} p(t_1-nT)p(t_2-nT).$$

b) For the randomly delayed version of the pulse-code-modulation process, we have the model

$$Y(t) = X(t-\Theta) = \sum_{m=-\infty}^{\infty} A_n p(t-nT-\Theta).$$

Therefore,

$$R_Y(t_1, t_2) = E\{Y(t_1)Y(t_2)\} = E\{E\{Y(t_1)Y(t_2) \mid \Theta\}\}$$

$$= E\{\sum_{n=-\infty}^{\infty} p(t_1-nT-\Theta)p(t_2-nT-\Theta)\}$$

$$= \sum_{n=-\infty}^{\infty} \frac{1}{T} \int_{-T/2}^{T/2} p(t_1-nT-\theta)p(t_2-nT-\theta)d\theta$$

$$= \frac{1}{T}\int_{-\infty}^{\infty} p(t_1-t_2+s)p(s)ds = \frac{1}{T} r_p(t_1-t_2).$$

4.12 The autocorrelation of the quadrature-amplitude-modulated process $Y(t)$ is given by

$$R_Y(t_1, t_2) = E\{Y(t_1)Y(t_2)\}$$

$$= E\{[X(t_1)\cos(\omega_0 t_1) - Z(t_1)\sin(\omega_0 t_1)][X(t_2)\cos(\omega_0 t_2) - Z(t_2)\sin(\omega_0 t_2)]\}$$

$$= E\{X(t_1)X(t_2)\}\cos(\omega_0 t_1)\cos(\omega_0 t_2) + E\{Z(t_1)Z(t_2)\}\sin(\omega t_1)\sin(\omega_0 t_2)$$

$$= R_X(t_1, t_2)\cos(\omega_0 t_1)\cos(\omega_0 t_2) + R_Z(t_1, t_2)\sin(\omega_0 t_1)\sin(\omega_0 t_2).$$

If $R_X(t_1, t_2) = R_Z(t_1, t_2) = R(t_1 - t_2)$, then we have

$$R_Y(t_1, t_2) = R(t_1 - t_2)[\cos(\omega_0 t_1)\cos(\omega_0 t_2) + \sin(\omega_0 t_1)\sin(\omega_0 t_2)]$$

$$= R(t_1 - t_2)\cos(\omega_0[t_1 - t_2]) = R_Y(t_1 - t_2).$$

4.13 Using + for Y_1 and − for Y_2, the autocorrelation functions are given by

$$R_Y(t_1, t_2) = E\{Y(t_1)Y(t_2)\}$$

$$= E\{[X(t_1) \pm Z(t_1)][X(t_2) \pm Z(t_2)]\}$$

$$= E\{X(t_1)X(t_2)\} + E\{Z(t_1)Z(t_2)\} \pm E\{X(t_1)Z(t_2)\} \pm E\{Z(t_1)X(t_2)\}$$

$$= R_X(t_1, t_2) + R_Z(t_1, t_2) \pm m_X(t_1)m_Z(t_2) \pm m_Z(t_1)m_X(t_2).$$

The cross-correlation function is given by

$$R_{Y_1Y_2}(t_1, t_2) = E\{Y_1(t_1)Y_2(t_2)\} = E\{[X(t_1) + Z(t_1)][X(t_2) - Z(t_2)]\}$$

$$= R_X(t_1, t_2) - R_Z(t_1, t_2) - m_X(t_1)m_Z(t_2) + m_Z(t_1)m_X(t_2).$$

When $R_X = R_Z = R$ and $m_X = m_Z = m$, then the preceding results simplify to

$$R_Y(t_1, t_2) = 2[R(t_1, t_2) \pm m(t_1)m(t_2)]$$

and

$$R_{Y_1Y_2}(t_1, t_2) = 0.$$

Thus, the sum and difference processes for two i.i.d. processes are orthogonal to each other and, since the difference process has zero mean, they are also uncorrelated with each other.

4.14 The autocorrelation of $Y(t) = X(t) - X(t-T)$ is given by

$$R_Y(t_1, t_2) = E\{Y(t_1)Y(t_2)\} = E\{[X(t_1) - X(t_1 - T)][X(t_2) - X(t_2 - T)]\}$$

$$= R_X(t_1, t_2) + R_X(t_1 - T, t_2 - T) - R_X(t_1, t_2 - T) - R_X(t_1 - T, t_2)$$

$$= 2R_X(t_1 - t_2) - R_X(t_1 - t_2 + T) - R_X(t_1 - t_2 - T).$$

4.15 *a*) For the pulse-position-modulated signal, we have the model

$$X(t) = \sum_{n=-\infty}^{\infty} p(t - nT - P_n),$$

where $p(t)$ is the zero-position pulse and $\{P_n\}$ are independent of each other and identically distributed.

b) We obtain the mean as follows:

$$m_X(t) = E\{\sum_{n=-\infty}^{\infty} p(t - nT - P_n)\}$$

$$= \int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} p(t - nT - u) f_P(u) du = \sum_{n=-\infty}^{\infty} \bar{p}(t - nT),$$

where

$$\bar{p}(t) \triangleq p(t) \otimes f_P(t).$$

Thus,

$$m_X(t+T) = \sum_{n=-\infty}^{\infty} \bar{p}(t + T - nT) = \sum_{m=-\infty}^{\infty} \bar{p}(t - mT) = m_X(t), \qquad m = n - 1$$

and $m_X(t)$ is, therefore, periodic. Furthermore, if

$$f_P(p) = \begin{cases} \dfrac{1}{T - \Delta}, & 0 \le p < T - \Delta \\ 0, & otherwise \end{cases}$$

and

$$p(t) = \begin{cases} 1, & 0 \le t < \Delta \\ 0, & otherwise, \end{cases}$$

then

$$\bar{p}(t) = \frac{1}{T - \Delta} \int_0^{T-\Delta} p(t - u) du = \frac{1}{T - \Delta} \int_{t-(T-\Delta)}^{t} p(u) du$$

$$= \begin{cases} \dfrac{1}{T - \Delta} \displaystyle\int_0^t du, & 0 \le t < \Delta \\ \dfrac{1}{T - \Delta} \displaystyle\int_0^{\Delta} du, & \Delta \le t < T - \Delta \\ \dfrac{1}{T - \Delta} \displaystyle\int_{t-(T-\Delta)}^{\Delta} du, & T - \Delta \le t < T \end{cases}$$

$$= \begin{cases} \dfrac{t}{T - \Delta}, & 0 \le t < \Delta \\ \dfrac{\Delta}{T - \Delta}, & 0 \le t < T - \Delta \\ \dfrac{T - t}{T - \Delta}, & T - \Delta \le t < T. \end{cases}$$

c) For the phase-randomized version of the signal, we have the model

$$X(t) = \sum_{n=-\infty}^{\infty} p(t - nT - P_n - \Theta),$$

where

$$f_\Theta(\theta) = \begin{cases} 1/T, & -T/2 \le \theta < T/2 \\ 0, & \textit{otherwise}. \end{cases}$$

Therefore, the autocorrelation function is given by

$$R_X(t+\tau, t) = E\{X(t+\tau)X(t)\} = E\{E\{X(t+\tau)X(t) \mid \Theta\}$$

$$= E\{E\{\sum_{n,m=-\infty}^{\infty} p(t+\tau-nT-P_n-\Theta)p(t-mT-P_m-\Theta) \mid \Theta\}\}$$

$$= E\{\sum_{n=-\infty}^{\infty} E\{p(t+\tau-nT-P_n-\Theta)p(t-nT-P_n-\Theta) \mid \Theta\}\}$$

$$+ E\{\sum_{r\neq 0}\sum_{n=-\infty}^{\infty} E\{p(t+\tau-nT-P_n-\Theta)p(t-nT-rT-P_{n+r}-\Theta) \mid \Theta\}\}$$

(using $r = m - n$).

We have

$$E\{\sum_{n=-\infty}^{\infty} E\{p(t+\tau-nT-P_n-\Theta)p(t-nT-P_n-\Theta) \mid \Theta\}\}$$

$$= E\{\sum_{n=-\infty}^{\infty} \frac{1}{T-\Delta}\int_0^{T-\Delta} p(t+\tau-nT-u-\Theta)p(t-nT-u-\Theta)du \mid \Theta\}$$

$$= \frac{1}{T-\Delta}\int_0^{T-\Delta} \sum_{n=-\infty}^{\infty} \frac{1}{T}\int_{-T/2}^{T/2} p(t+\tau-nT-u-\theta)p(t-nT-u-\theta)d\theta du$$

$$= \frac{1}{T-\Delta}\int_0^{T-\Delta} \frac{1}{T}\int_{-\infty}^{\infty} p(v+\tau)p(v)dvdu = \frac{1}{T}r_p(\tau),$$

and we have

$$E\{\sum_{r\neq 0}\sum_{n=-\infty}^{\infty} E\{p(t+\tau-nT-P_n-\Theta)p(t-nT-rT-P_{n+r}-\Theta)\mid\Theta\}\}$$

$$=E\{\sum_{r\neq 0}\sum_{n=-\infty}^{\infty}\int_{-\infty}^{\infty} p(t+\tau-nT-u-\Theta)f_P(u)du\int_{-\infty}^{\infty} p(t-nT-rT-v-\Theta)f_P(v)dv\}$$

$$=\sum_{r\neq 0}\iint_{-\infty}^{\infty} f_P(u)f_P(v)[\sum_{n=-\infty}^{\infty}\frac{1}{T}\int_{-T/2}^{T/2} p(t+\tau-nT-u-\theta)p(t-nT-rT-v-\theta)d\theta]dudv$$

$$=\sum_{r\neq 0}\iint_{-\infty}^{\infty} f_P(u)f_P(v)\ \frac{1}{T}\int_{-\infty}^{\infty} p(s+\tau+v-u+rT)p(s)ds\ dudv$$

$$=\frac{1}{T}\sum_{r\neq 0}\iint_{-\infty}^{\infty} f_P(u)f_P(v)r_p(\tau+rT+v-u)dudv$$

$$=\frac{1}{T}\sum_{r\neq 0}\int_{-\infty}^{\infty} r_p(\tau+rT+z)\int_{-\infty}^{\infty} f_P(u)f_P(z+u)dudz$$

$$=\frac{1}{T}\sum_{r\neq 0}\int_{-\infty}^{\infty} r_p(\tau+rT+z)r_f(z)dz$$

$$=\frac{1}{T}\sum_{r\neq 0}\int_{-\infty}^{\infty} r_p(\tau+rT-z)r_f(z)dz \qquad \text{(since } r_f(\cdot) \text{ is even)}$$

$$=\frac{1}{T}\sum_{r\neq 0} r_p(\tau+rT)\otimes r_f(\tau+rT)=\frac{1}{T}\sum_{n\neq 0} r_p(\tau-nT)\otimes r_f(\tau-nT).$$

Therefore, we have the desired result (4.43).

4.16 *a*) For the pulse-width-modulated signal, we have the model

$$X(t)=\sum_{n=-\infty}^{\infty} p\left(\frac{t-nT}{W_n}\right),$$

where

$$p(t)\triangleq\begin{cases}1, & 0\le t<1\\ 0, & otherwise\end{cases}$$

and $\{W_n\}$ are independent of each other and identically distributed.

b) The mean of $X(t)$ is given by

$$m_X(t)=\int_{-\infty}^{\infty}\sum_{n=-\infty}^{\infty} p\left(\frac{t-nT}{w}\right)f_W(w)dw=\sum_{n=-\infty}^{\infty} q(t-nT),$$

where

$$q(t) \triangleq \int_{-\infty}^{\infty} p(t/w) f_W(w) dw.$$

Also,

$$m_X(t+T) = \sum_{n=-\infty}^{\infty} q(t+T-nT) = \sum_{m=-\infty}^{\infty} q(t-mT) = m_X(t), \quad m = n-1.$$

Thus, $m_X(t)$ is periodic.

c) For the phase-randomized version of the signal, we have the model

$$X(t) = \sum_{n=-\infty}^{\infty} p\left(\frac{t-nT-\Theta}{W_n}\right).$$

Therefore, the autocorrelation of $X(t)$ is given by

$$\begin{aligned}
R_X(t+\tau, t) &= E\{E\{X(t+\tau)X(t) \mid \Theta\}\} \\
&= E\{E\{\sum_{n,m=-\infty}^{\infty} p\left(\frac{t+\tau-nT-\Theta}{W_n}\right) p\left(\frac{t-mT-\Theta}{W_m}\right) \mid \Theta\}\} \\
&= E\{\sum_{n=-\infty}^{\infty} E\{p\left(\frac{t+\tau-nT-\Theta}{W_n}\right) p\left(\frac{t-nT-\Theta}{W_n}\right) \mid \Theta\}\} \\
&\quad + E\{\sum_{r\neq 0} \sum_{n=-\infty}^{\infty} E\{p\left(\frac{t+\tau-nT-\Theta}{W_n}\right) p\left(\frac{t-nT-rT-\Theta}{W_{n+r}}\right) \mid \Theta\}\}
\end{aligned}$$

(using $r = m-n$).

We have

$$\begin{aligned}
&E\{\sum_{n=-\infty}^{\infty} E\{p\left(\frac{t+\tau-nT-\Theta}{W_n}\right) p\left(\frac{t-nT-\Theta}{W_n}\right) \mid \Theta\}\} \\
&= E\{\sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} p\left(\frac{t+\tau-nT-\Theta}{w}\right) p\left(\frac{t-nT-\Theta}{w}\right) f_W(w) dw\} \\
&= \int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \frac{1}{T} \int_{-T/2}^{T/2} p\left(\frac{t+\tau-nT-\theta}{w}\right) p\left(\frac{t-nT-\theta}{w}\right) d\theta \, f_W(w) dw \\
&= \frac{1}{T} \iint_{-\infty}^{\infty} p\left(\frac{\phi+\tau}{w}\right) p\left(\frac{\phi}{w}\right) d\phi \, f_W(w) dw \\
&= \frac{1}{T} \int_{-\infty}^{\infty} w \int_{-\infty}^{\infty} p(\psi+\tau/w) p(\psi) d\psi \, f_W(w) dw, \quad \psi = \phi/w \\
&= \frac{1}{T} \int_{-\infty}^{\infty} r_p\left(\frac{\tau}{w}\right) w f_W(w) dw = \frac{1}{T} E\{r_p\left(\frac{\tau}{W}\right) W\},
\end{aligned}$$

and we have

$$E\{\sum_{r\neq 0}\sum_{n=-\infty}^{\infty} E\{p(\frac{t+\tau-nT-\Theta}{W_n})p(\frac{t-nT-rT-\Theta}{W_{n+r}})\mid\Theta\}\}$$

$$=E\{\sum_{r\neq 0}\sum_{n=-\infty}^{\infty}\int_{-\infty}^{\infty} p(\frac{t+\tau-nT-\Theta}{u})f_W(u)du\int_{-\infty}^{\infty} p(\frac{t-nT-rT-\Theta}{v})f_W(v)dv\}$$

$$=\sum_{r\neq 0}\iint_{-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\frac{1}{T}\int_{-T/2}^{T/2} p(\frac{t+\tau-nT-\theta}{u})p(\frac{t-nT-rT-\theta}{v})d\theta f_W(u)f_W(v)dudv$$

$$=\frac{1}{T}\sum_{r\neq 0}\iint_{-\infty}^{\infty} f_W(u)f_W(v)\int_{-\infty}^{\infty} p(\frac{s+\tau+rT}{u})p(\frac{s}{v})ds\ dudv$$

$$=\frac{1}{T}\sum_{r\neq 0}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p(\frac{s+\tau+rT}{u})f_W(u)du\int_{-\infty}^{\infty} p(\frac{s}{v})f_W(v)dv\ ds$$

$$=\frac{1}{T}\sum_{r\neq 0}\int_{-\infty}^{\infty} q(s+\tau+rT)q(s)ds,\qquad q(t)=E\{p(t/W)\}$$

$$=\frac{1}{T}\sum_{r\neq 0} r_q(\tau+rT)=\frac{1}{T}\sum_{n\neq 0} r_q(\tau-nT).$$

Therefore, we have the desired result (4.45).

Chapter 5

Classes of Random Processes

5.1 A random-amplitude sine wave can be modeled as

$$X(t) = A\cos(\omega_0 t + \theta),$$

where A is a Gaussian random variable.

a) Consider the linear combination

$$Z = \sum_{i=1}^{n} a_i X(t_i) = Ac,$$

where

$$c = \sum_{i=1}^{n} a_i \cos(\omega_0 t_i + \theta),$$

which is a non-random constant. Since A is a Gaussian random variable, then Z is a Gaussian random variable for all n, $\{a_i\}_1^n$ and $\{t_i\}_1^n$. Thus, $X(t)$ is a Gaussian process.

b) Let

$$\mathbf{X} = [X(t_1) \quad X(t_2) \cdots X(t_n)]^T$$
$$= A[\cos(\omega_0 t_1 + \theta) \quad \cos(\omega_0 t_2 + \theta) \cdots \cos(\omega_0 t_n + \theta)]^T \triangleq A\,\mathbf{r}.$$

Then

$$\mathbf{K_X} = E\{[\mathbf{X} - E\{\mathbf{X}\}][\mathbf{X} - E\{\mathbf{X}\}]^T\} = \sigma_A^2 \mathbf{r}\mathbf{r}^T.$$

Since for every vector $\mathbf{a}$, the vector

$$\mathbf{b} = \mathbf{K_X}\mathbf{a} = d\,\mathbf{r},$$

where d is a constant, is in a space of dimension one (spanned by $\mathbf{r}$), then the range space of $\mathbf{K_X}$ has dimension equal to unity, and, therefore, $\mathbf{K_X}$ has rank equal to unity.

c) At t_1 and t_2, we have (ignoring the time points at which $X(t) = 0$)

$$X(t_1) = A\cos(\omega_0 t_1 + \theta) = c_1 A$$

$$X(t_2) = A\cos(\omega_0 t_2 + \theta) = \frac{c_2}{c_1} X(t_1).$$

Thus, we obtain from (1.34)

$$f_{X(t_1)X(t_2)}(x_1, x_2) = \frac{1}{c_1 \sigma_A \sqrt{2\pi}} \exp\{-\frac{1}{2}[\frac{x_1 - m_A c_1}{\sigma_A c_1}]^2\}\,\delta(x_2 - \frac{c_2}{c_1} x_1).$$

5.2 Consider the linear combination

$$Z = \sum_{i=1}^{n} a_i X(t_i) = \sum_{i=1}^{n} \sum_{m=-\infty}^{\infty} a_i V(mT) h(t_i - mT) = \sum_{m=-\infty}^{\infty} c_m V(mT),$$

where

$$c_m \triangleq \sum_{i=1}^{n} a_i h(t_i - mT),$$

which is a nonrandom constant for each m. Thus, Z is a linear combination of jointly Gaussian random variables $\{V(mT)\}$ and is, therefore, a Gaussian random variable. Thus $X(t)$ is a Gaussian random process.

5.3 *a*) For $r = 4$, we obtain from (5.58)

$$E\{X_1X_2X_3X_4\} = E\{X_1X_2\}E\{X_3X_4\} + E\{X_1X_3\}E\{X_2X_4\} + E\{X_1X_4\}E\{X_2X_3\}.$$

Now, let $X_i = X(t_i)$, $i=1, \ldots, 4$; then from the above we obtain (5.59) since $E\{X(t_i)X(t_j)\} = K_X(t_i, t_j)$ for a zero-mean process.

b) For $n = 4$, from (5.56) and (2.38) we have

$$\Phi(\omega_1, \ldots, \omega_4) = \exp\{-\frac{1}{2} \sum_{i,j=1}^{4} \omega_i \omega_j K_X(t_i, t_j)\}.$$

For $k_i = 1$, $i=1, \ldots, 4$ in the left member of (5.57), we obtain

$$\frac{\partial \Phi(\omega_1, \ldots, \omega_4)}{\partial \omega_1} = [-\frac{1}{2} \sum_{i=1}^{4} \omega_i K_X(t_i, t_1) - \frac{1}{2} \sum_{j=1}^{4} \omega_j K_X(t_1, t_j)] \Phi(\omega_1, \ldots, \omega_4)$$

$$= -\sum_{j=1}^{4} \omega_j K_X(t_1, t_j) \Phi(\omega_1, \ldots, \omega_4)$$

$$\frac{\partial^2 \Phi(\omega_1, \ldots, \omega_4)}{\partial \omega_1 \partial \omega_2} = -K_X(t_1, t_2)\Phi(\omega_1, \ldots, \omega_4)$$

$$+ \sum_{j=1}^{4} \omega_j K_X(t_1, t_j) \sum_{i=1}^{4} \omega_i K_X(t_2, t_i) \Phi(\omega_1, \ldots, \omega_4)$$

$$\frac{\partial^3 \Phi(\omega_1, \ldots, \omega_4)}{\partial \omega_1 \partial \omega_2 \partial \omega_3} = K_X(t_1, t_2) \sum_{j=1}^{4} \omega_j K_X(t_3, t_j) \Phi(\omega_1, \ldots, \omega_4)$$

$$+ K_X(t_1, t_3) \sum_{i=1}^{4} \omega_i K_X(t_2, t_i) \Phi(\omega_1, \ldots, \omega_4)$$

$$+ K_X(t_2, t_3) \sum_{j=1}^{4} \omega_j K_X(t_1, t_j) \Phi(\omega_1, \ldots, \omega_4)$$

$$\frac{\partial^4 \Phi(\omega_1, \ldots, \omega_4)}{\partial \omega_1 \partial \omega_2 \partial \omega_3 \partial \omega_4} = K_X(t_1, t_2) K_X(t_3, t_4) \Phi(\omega_1, \ldots, \omega_4)$$

$$- K_X(t_1, t_2)\sum_{j=1}^{4} \omega_j K_X(t_3, t_j)\sum_{i=1}^{4} \omega_i K_X(t_4, t_i)\Phi(\omega_1, \ldots, \omega_4)$$

$$+ K_X(t_1, t_3)K_X(t_2, t_4)\Phi(\omega_1, \ldots, \omega_4)$$

$$- K_X(t_1, t_3)\sum_{i=1}^{4} \omega_i K_X(t_2, t_i)\sum_{j=1}^{4} \omega_j K_X(t_4, t_j)\Phi(\omega_1, \ldots, \omega_4)$$

$$+ K_X(t_2, t_3)K_X(t_1, t_4)\Phi(\omega_1, \ldots, \omega_4)$$

$$- K_X(t_2, t_3)\sum_{j=1}^{4} \omega_j K_X(t_1, t_j)\sum_{i=1}^{4} \omega_i K_X(t_4, t_i)\Phi(\omega_1, \ldots, \omega_4).$$

Thus, (5.57) yields

$$E\{X(t_1)X(t_2)X(t_3)X(t_4)\} = m(1, 1, 1, 1) = \frac{1}{i^4}\left.\frac{\partial^4\Phi(\omega_1, \ldots, \omega_4)}{\partial\omega_1\partial\omega_2\partial\omega_3\partial\omega_4}\right|_{\omega_1=\omega_2=\omega_3=\omega_4=0}$$

$$= K_X(t_1, t_2)K_X(t_3, t_4) + K_X(t_1, t_3)K_X(t_2, t_4) + K_X(t_1, t_4)K_X(t_2, t_3),$$

since $\Phi(0, 0, 0, 0) = 1$.

5.4 We denote $X(t_i)$ by X_i for $i = 1, \ldots, 4$ and

$$K_{ij} \triangleq E\{(X_i - m_i)(X_j - m_j)\} = R_{ij} - m_i m_j,$$

where

$$R_{ij} \triangleq E\{X_i X_j\} \quad \text{and} \quad m_i \triangleq E\{X_i\}.$$

Since the third-order moment for zero-mean jointly Gaussian random variables are zero, $E\{(X_i - m_i)(X_j - m_j)(X_k - m_k)\} = 0$, then

$$E\{X_i X_j X_k\} = m_i R_{jk} + m_j R_{ik} + m_k R_{ij} - 2m_i m_j m_k. \tag{*}$$

The fourth-order moments for zero-mean jointly Gaussian random variables are given by

$$E\{(X_i - m_i)(X_j - m_j)(X_k - m_k)(X_l - m_l)\} = E\{X_i(X_j - m_j)(X_k - m_k)(X_l - m_l)\} \tag{**}$$

$$= E\{X_i X_j X_k X_l\} - m_l E\{X_i X_j X_k\} - m_k E\{X_i X_j X_l\} - m_j E\{X_i X_k X_l\}$$

$$+ m_k m_l R_{ij} + m_j m_l R_{ik} + m_l m_k R_{ij} - m_i m_j m_k m_l$$

$$= E\{X_i X_j X_k X_l\} - m_l m_i R_{jk} - m_l m_j R_{ik} - m_l m_k R_{ij}$$

$$- m_k m_i R_{jl} - m_k m_j R_{il} - m_j m_i R_{kl} + 5m_i m_j m_k m_l.$$

From exercise 5.3, we know that (**) is also equal to

$$(R_{ij}-m_im_j)(R_{kl}-m_km_l)+(R_{ik}-m_im_k)(R_{jl}-m_jm_l)+(R_{il}-m_im_l)(R_{jk}-m_jm_k)$$
$$=R_{ij}R_{kl}+R_{ik}R_{jl}+R_{il}R_{jk}-m_im_jR_{kl}-m_km_lR_{ij}$$
$$-m_jm_lR_{ik}-m_im_kR_{jl}-m_jm_kR_{il}-m_im_lR_{jk}+3m_im_jm_km_l.$$

Thus, we have

$$E\{X_iX_jX_kX_l\}=R_{ij}R_{kl}+R_{ik}R_{jl}+R_{il}R_{jk}-2m_im_jm_km_l.$$

5.5 The mean of $Y(t)=X(t)X(t-\Delta)$ is

$$m_Y=E\{X(t)X(t-\Delta)\}=R_X(\Delta)=K_X(\Delta)+m_X^2.$$

Using the result from exercise 5.4, we obtain the autocorrelation for $Y(t)$,

$$R_Y(\tau)=E\{Y(t+\tau)Y(t)\}=E\{X(t+\tau)X(t+\tau-\Delta)X(t)X(t-\Delta)\}$$
$$=E\{X(t+\tau)X(t+\tau-\Delta)\}E\{X(t)X(t-\Delta)\}+E\{X(t+\tau)X(t)\}E\{X(t+\tau-\Delta)X(t-\Delta)\}$$
$$+E\{X(t+\tau)X(t-\Delta)\}E\{X(t+\tau-\Delta)X(t)\}$$
$$-2E\{X(t+\tau)\}E\{X(t+\tau-\Delta)\}E\{X(t)\}E\{X(t-\Delta)\}$$
$$=R_X^2(\Delta)+R_X^2(\tau)+R_X(\tau+\Delta)R_X(\tau-\Delta)-2m_X^4.$$

5.6 Let $Z(t)=X(t)$, then $Y(t)=X(t)Z(t)=X^2(t)$ and, from (5.59), we obtain

$$R_Y(t_1,t_2)=E\{Y(t_1)Y(t_2)\}=E\{X^2(t_1)X^2(t_2)\}$$
$$=K_X(t_1,t_1)K_X(t_2,t_2)+2K_X^2(t_1,t_2)$$
$$=R_X(t_1,t_1)R_X(t_2,t_2)+2R_X^2(t_1,t_2).$$

From exercise 4.9 where $X(t)$ and $Z(t)$ are independent, we obtain

$$R_Y(t_1,t_2)=R_X(t_1,t_2)R_Z(t_1,t_2)=R_X^2(t_1,t_2).$$

This illustrates that the result in exercise 4.9 is invalid when $X(t)$ and $Z(t)$ are dependent.

5.7 *a*) Both phase-randomized processes in Sections 4.2.4 and 4.2.6 can be put into the form $Y(t)=X(t+\Theta)$, where Θ is uniformly distributed over $(-T/2,T/2]$. Thus, the definition of conditional probability density yields

$$f_{Y(t)}(y)=\int_{-\infty}^{\infty}f_{Y(t)|\Theta}(y|\theta)f_\Theta(\theta)d\theta=\int_{-\infty}^{\infty}f_{X(t+\theta)}(y)f_\Theta(\theta)d\theta$$
$$=\frac{1}{T}\int_{-T/2}^{T/2}f_{X(t+\theta)}(y)d\theta.$$

Hence, if $X(t)$ is Gaussian and nonstationary so that the particular Gaussian probability density $f_{X(t+\theta)}$ varies in θ over at least some subintervals of $(-T/2, T/2]$, then $f_{Y(t)}$ is a (continuous) uniform additive mixture of non-identical Gaussian functions and, therefore, cannot itself be Gaussian. For example, it is easy to show that $\frac{1}{2} f_{X(t+\theta_1)} + \frac{1}{2} f_{X(t+\theta_2)}$ cannot be Gaussian unless $f_{X(t+\theta_1)} = f_{X(t+\theta_2)}$.

b) Since

$$Y(t) = A\sin(\omega_0 t + \Theta) = A\sin(\omega_0 t)\cos(\Theta) + A\cos(\omega_0 t)\sin(\Theta)$$
$$= W\sin(\omega_0 t) + Z\cos(\omega_0 t),$$

where W and Z are i.i.d. Gaussian random variables and are therefore jointly Gaussian, then we know from the definition of jointly Gaussian random variables that $Y(t)$ is a Gaussian random variable for each value of t. To prove that $Y(t)$ is a Gaussian *process*, we must prove that

$$X \triangleq \sum_{i=1}^{n} a_i Y(t_i)$$

is a Gaussian random variable for every n, every set of n times $\{t_i\}_1^n$ and every set of n real numbers $\{a_i\}_1^n$. Substitution of the expression for $Y(t)$ into the definition of X yields

$$X = \alpha W + \beta Z,$$

where

$$\alpha = \sum_{i=1}^{n} a_i \sin(\omega_0 t_i) \quad \text{and} \quad \beta = \sum_{i=1}^{n} a_i \cos(\omega_0 t_i).$$

Since we know that W and Z are jointly Gaussian, then clearly X is a Gaussian random variable. Therefore, $Y(t)$ is a Gaussian process.

5.8 From exercise 1.12*d* we know that if X and Y are jointly Gaussian, then $X|Y$ is Gaussian with parameters α'' and β'' given by

$$\beta'' = \beta + \frac{\gamma(y - \beta')\alpha}{\alpha'}$$
$$\alpha'' = \alpha\sqrt{1-\gamma^2}.$$

Let $Y = X(t_1)$ and $X = X(t_2)$, and

$$\beta = E\{X\} = E\{X(t_2)\} = m_X(t_2)$$
$$\alpha^2 = E\{(X-\beta)^2\} = E\{[X(t_2) - m_X(t_2)]^2\} = \sigma_X^2(t_2)$$
$$\beta' = E\{Y\} = E\{X(t_1)\} = m_X(t_1)$$

$$\alpha'^2 = E\{(Y-\beta')^2\} = E\{[X(t_1)-m_X(t_1)]^2\} = \sigma_X^2(t_1)$$

$$\gamma = \frac{E\{(X-\beta)(Y-\beta')\}}{\alpha\alpha'} = \frac{E\{[X(t_2)-m_X(t_2)][X(t_1)-m_X(t_1)]\}}{\sigma_X(t_2)\sigma_X(t_1)}$$

$$= \frac{K_X(t_1, t_2)}{\sigma_X(t_1)\sigma_X(t_2)};$$

then

$$E\{X(t_2)|X(t_1)\} = E\{X|Y\} = \beta'' = m_X(t_2) + \frac{K_X(t_1, t_2)}{\sigma_X^2(t_1)}[X(t_1)-m_X(t_1)].$$

5.9 Since $dg(X)/dX = 2\delta(X)$, then

$$E\{\frac{dg(X_1)}{dX_1}\frac{dg(X_2)}{dX_2}\} = \iint_{-\infty}^{\infty} 4\delta(x_1)\delta(x_2)f_{X_1X_2}(x_1, x_2)dx_1dx_2$$

$$= 4f_{X_1X_2}(0, 0) = \frac{4}{2\pi\sigma_X^2\sqrt{1-\rho^2}},$$

where $f_{X_1X_2}(x_1, x_2)$ is the bivariate Gaussian density (2.27) with zero means and equal variances. Therefore, (5.61) yields

$$\frac{dR_Y}{d\rho} = \frac{2}{\pi}\frac{1}{\sqrt{1-\rho^2}}.$$

Integrating this result with respect to ρ yields

$$R_Y = \frac{2}{\pi}\sin^{-1}(\rho).$$

Since

$$g(x) = (\frac{2}{\pi\alpha^2})^{1/2}\int_0^x \exp\{-z^2/2\alpha^2\}dz$$

then

$$\frac{dg(x)}{dx} = (\frac{2}{\pi\alpha^2})^{1/2}\exp\{-x^2/2\alpha^2\}.$$

Applying (1.36) for $Y = g(X)$ yields

$$f_Y(y) = \frac{f_X(x)}{|dg(x)/dx|} = \frac{1}{\sqrt{2\pi R_X(0)}}\exp\{-x^2/2R_X(0)\} \div (\frac{2}{\pi\alpha^2})^{1/2}\exp\{-x^2/2\alpha^2\}$$

$$= \frac{1}{2}\frac{\alpha}{\sqrt{R_X(0)}}\exp\{-\frac{x^2}{2}(\frac{1}{R_X(0)}-\frac{1}{\alpha^2})\} = \frac{1}{2} \quad \text{for} \quad \alpha^2 = R_X(0).$$

Also, since $g(\infty) = 1$, $g(-\infty) = -1$, and $g(\cdot)$ is a monotonically increasing function, then the range of $Y = g(X)$ is $|Y| < 1$. Thus, we have

$$f_Y(y) = \begin{cases} 1/2, & |y| < 1 \\ 0, & \textit{otherwise.} \end{cases}$$

5.10 *a*) Given

$$g(X) = \begin{cases} +1, & X \geq 0 \\ -1, & X < 0, \end{cases}$$

we obtain $dg/dX = 2\delta(X)$. Letting $X_1 = X(t+\tau)$, $X_2 = X(t)$, $Y_1 = X_1$, $Y_2 = g(X_2)$, and applying Price's theorem (exercise 5.9), we obtain

$$\frac{dR_Y}{d\rho} = R_X(0)E\{(1)2\delta(X_2)\}$$

$$= R_X(0)2\int_{-\infty}^{\infty} \delta(x)\frac{1}{\sqrt{2\pi R_X(0)}}\exp\{-\frac{x^2}{2R_X(0)}\}dx = \sqrt{2R_X(0)/\pi}.$$

Thus,

$$R_Y = \int \sqrt{2R_X(0)/\pi}\, d\rho = \rho\sqrt{2R_X(0)/\pi} = \frac{R_X(\tau)}{R_X(0)}\sqrt{2R_X(0)/\pi}.$$

Hence,

$$E\{X(t+\tau)g[X(t)]\} = cR_X(\tau),$$

where $c = \sqrt{2/\pi R_X(0)}$.

b) For an arbitrary nonlinearity $g(\cdot)$, the method of part *a* yields

$$\frac{dR_Y}{d\rho} = R_X(0)E\{\frac{dg[X(t)]}{dX(t)}\} = cR_X(0),$$

where $c = E\{dg[X(t)]/dX(t)\}$. Thus,

$$E\{X(t+\tau)g[X(t)]\} = R_Y = cR_X(0)\rho = cR_X(\tau).$$

5.11 Let X_{n+1} be the position at time $n+1$ and Z_n be the independent random error at time n. Then, we have

$$X_{n+1} = aX_n + Z_n.$$

Thus, since Z_n is independent of X_i, $i = n, n-1, \ldots, 1$, we have

$$f_{X_{n+1}|X_n \ldots X_1}(x_{n+1}|x_n, \ldots, x_1) = f_{Z_n}(x_{n+1} - ax_n) = f_{X_{n+1}|X_n}(x_{n+1}|x_n).$$

Hence X_n is Markov process.

5.12 Let $P_{10} = P_{01} = r$. Then $P_{00} = P_{11} = 1 - r = q$ and

$$\mathbf{P} = \begin{bmatrix} q & r \\ r & q \end{bmatrix}$$

The eigenvalues and eigenvectors of $\mathbf{P}$ are found to be

$$\lambda_1 = 1, \qquad \mathbf{x}_1 = [\,1/\sqrt{2} \quad 1/\sqrt{2}\,]^T$$
$$\lambda_2 = 1-2r, \qquad \mathbf{x}_2 = [\,1/\sqrt{2} \quad -1/\sqrt{2}\,]^T,$$

and $\mathbf{P}$ can be expressed in terms of the singular value decomposition

$$\mathbf{P} = \begin{bmatrix} \mathbf{x}_1 & \mathbf{x}_2 \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} \mathbf{x}_1^T \\ \mathbf{x}_2^T \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1-2r \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$$

Furthermore, $\mathbf{P}^n$ can be expressed as

$$\mathbf{P}^n = \begin{bmatrix} \mathbf{x}_1 & \mathbf{x}_2 \end{bmatrix} \begin{bmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{bmatrix} \begin{bmatrix} \mathbf{x}_1^T \\ \mathbf{x}_2^T \end{bmatrix} = \lambda_1^n\, \mathbf{x}_1\mathbf{x}_1^T + \lambda_2^n\, \mathbf{x}_2\mathbf{x}_2^T.$$

Thus, (5.13) becomes

$$\mathbf{P}_n = \mathbf{P}^n \mathbf{P}_0 = (\lambda_1^n\, \mathbf{x}_1\mathbf{x}_1^T + \lambda_2^n\, \mathbf{x}_2\mathbf{x}_2^T)\mathbf{P}_0.$$

Hence, when $r = 1/2$, then $\lambda_2 = 0$ and, therefore,

$$\mathbf{P}_n = \mathbf{x}_1\mathbf{x}_1^T\mathbf{P}_0 = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix} \mathbf{P}_0 \quad \text{for all } n > 0;$$

Thus, the state is stationary. When $r > 1/2$, then $-1 < \lambda_2 < 0$ and λ_2^n therefore alternates in sign and approaches zero. When $r < 1/2$, then $0 < \lambda_2 < 1$ and λ_2^n is therefore positive but still approaches zero. Hence, for $r \neq 1/2$,

$$\mathbf{P}_n \to \mathbf{x}_1\mathbf{x}_1^T\mathbf{P}_0 = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix} \mathbf{P}_0.$$

Thus, the state is asymptotically stationary. Observe that since the elements of $\mathbf{P}_0$ are nonnegative, then $\mathbf{P}_0$ cannot be orthogonal to $\mathbf{x}_1$. However, if $P_{X_0}(1) = P_{X_0}(0)$, then $\mathbf{P}_0$ is orthogonal to $\mathbf{x}_2$. In this case, we see that

$$\mathbf{P}_n = (\lambda_1^n\, \mathbf{x}_1\mathbf{x}_1^T + \lambda_2^n\, \mathbf{x}_2\mathbf{x}_2^T)\mathbf{P}_0 = \lambda_1^n\, \mathbf{x}_1\mathbf{x}_1^T\mathbf{P}_0 \quad \text{for all} \quad n > 0,$$

regardless of the value of r. Thus, this particular initial state distribution renders the state stationary regardless of the state transition probability r.

5.13 We have the model

$$X_n = X_{n-1} + Z_{n-1} + W_{n-1},$$

where

$$W_{n-1} = \begin{cases} 0, & \text{previous service incomplete, } prob = 1 - p \\ -1, & \text{previous service complete, } prob = p. \end{cases}$$

Consequently,

$$\begin{aligned} &Prob\{X_n = k_n | X_{n-1} = k_{n-1}, X_{n-2} = k_{n-2}, \ldots, X_1 = k_1\} \\ &\qquad = Prob\{[(Z_{n-1} = k_n - k_{n-1}) \cap (W_{n-1} = 0)] \\ &\qquad\quad \cup [(Z_{n-1} = k_n - k_{n-1} + 1) \cap (W_{n-1} = -1)]\}. \end{aligned}$$

Since this probability does not depend on $k_{n-2}, k_{n-3}, \ldots, k_1$, then X_n is a Markov process. We can obtain a more explicit expression for this transition probability as follows:

$$\begin{aligned} Prob\{X_n = k_n | X_{n-1} = k_{n-1}\} &= P_Z(k_n - k_{n-1})P_W(0) + P_Z(k_n - k_{n-1} + 1)P_W(-1) \\ &= (1-p)\frac{\lambda^{k_n - k_{n-1}}}{(k_n - k_{n-1})!}e^{-\lambda} + p\frac{\lambda^{k_n - k_{n-1}+1}}{(k_n - k_{n-1}+1)!}e^{-\lambda} \\ &= \frac{\lambda^{k_n - k_{n-1}}}{(k_n - k_{n-1})!}e^{-\lambda}\left(1 - p + \frac{\lambda p}{k_n - k_{n-1} + 1}\right). \end{aligned}$$

5.14 During the $(n-1)$st service period, the device in service will either fail or not; therefore, given the age, $X_{n-1} = k_{n-1}$, of the device in service at time $n-1$, the age of the device in service at time n is one of two values:

$$(X_n | X_{n-1} = k_{n-1}) = \begin{cases} k_{n-1} + 1, & prob = 1 - p_{k_{n-1}} \\ 1, & prob = p_{k_{n-1}}. \end{cases}$$

It is clear from the problem statement that knowing the ages $X_{n-2}, X_{n-3}, \ldots X_1$ at earlier times will have no effect on this conditional probability distribution. Thus, this is a Markov process. The transition probabilities are

$$P(X_n = k_n | X_{n-1} = k_{n-1}) = \begin{cases} 1 - p_{k_{n-1}}, & k_n = k_{n-1} + 1 \\ p_{k_{n-1}}, & k_n = 1 \\ 0, & otherwise. \end{cases}$$

5.15 *a*) From (5.6) we obtain

$$\begin{aligned} &\int_{-\infty}^{\infty} f_{X(t_3)|X(t_2)}(x_3|x_2) f_{X(t_2)|X(t_1)}(x_2|x_1)dx_2 \\ &\qquad = \int_{-\infty}^{\infty} f_{X(t_3)|X(t_2)X(t_1)}(x_3|x_2, x_1) f_{X(t_2)|X(t_1)}(x_2|x_1)dx_2. \end{aligned}$$

But then by using the definition of conditional probability density and $X(t_3)|X(t_2)X(t_1) = [X(t_3)|X(t_2)]|X(t_1)$, this latter expression becomes

$$\int_{-\infty}^{\infty} f_{X(t_3)X(t_2)|X(t_1)}(x_3, x_2|x_1)dx_2 = f_{X(t_3)|X(t_1)}(x_3|x_1).$$

This verifies (5.9). The discrete-distribution counterpart of (5.9) is given by

$$P_{X(t_3)|X(t_1)}(x_3|x_1) = \sum_{x_2} P_{X(t_3)|X(t_2)}(x_3|x_2)\, P_{X(t_2)|X(t_1)}(x_2|x_1).$$

b) Repeated use of the procedure applied in part *a* yields

$$\int_{-\infty}^{\infty} \cdots \int f_{X_n|X_{n-1}}(x_n|x_{n-1}) f_{X_{n-1}|X_{n-2}}(x_{n-1}|x_{n-2}) f_{X_{n-2}|X_{n-3}}(x_{n-2}|x_{n-3}) \times \cdots$$

$$\times f_{X_{m+1}|X_m}(x_{m+1}|x_m)dx_{n-1} \cdots dx_{m+1}$$

$$= \int_{-\infty}^{\infty} \cdots \int [\int_{-\infty}^{\infty} f_{X_n|X_{n-1}X_{n-2}}(x_n|x_{n-1}, x_{n-2}) f_{X_{n-1}|X_{n-2}}(x_{n-1}|x_{n-2})dx_{n-1}]$$

$$\times f_{X_{n-2}|X_{n-3}}(x_{n-2}|x_{n-3}) \cdots f_{X_{m+1}|X_m}(x_{m+1}|x_m)dx_{n-2} \cdots dx_{m+1}$$

$$= \int_{-\infty}^{\infty} \cdots \int [\int_{-\infty}^{\infty} f_{X_nX_{n-1}|X_{n-2}}(x_n, x_{n-1}|x_{n-2})dx_{n-1}] f_{X_{n-2}|X_{n-3}}(x_{n-2}|x_{n-3}) \times \cdots$$

$$\times f_{X_{m+1}|X_m}(x_{m+1}|x_m)dx_{n-2} \cdots dx_{m+1}.$$

$$= \int_{-\infty}^{\infty} \cdots \int f_{X_n|X_{n-2}}(x_n|x_{n-2}) f_{X_{n-2}|X_{n-3}}(x_{n-2}|x_{n-3}) \times \cdots$$

$$\times f_{X_{m+1}|X_m}(x_{m+1}|x_m)dx_{n-2} \cdots dx_{m+1}$$

$$\vdots$$

$$= f_{X_n|X_m}(x_n|x_m).$$

5.16 *a*) From Figure 5.2, we obtain

$$P_n(1) = P_{11}P_{X_{n-1}}(1) + P_{10}P_{X_{n-1}}(0)$$
$$P_n(0) = P_{01}P_{X_{n-1}}(1) + P_{00}P_{X_{n-1}}(0)$$

or

$$\mathbf{P}_n = \mathbf{P}\mathbf{P}_{n-1},$$

where

$$\mathbf{P}_n \triangleq \begin{bmatrix} P_{X_n}(1) \\ P_{X_n}(0) \end{bmatrix} \quad \text{and} \quad \mathbf{P} = \begin{bmatrix} P_{11} & P_{01} \\ P_{10} & P_{00} \end{bmatrix}$$

Given the initial state probabilities $\mathbf{P}_0$, the above can be reexpressed as

$$\mathbf{P}_n = \mathbf{P}\mathbf{P}_{n-1} = \mathbf{P}^2\mathbf{P}_{n-2} = \ldots = \mathbf{P}^n\mathbf{P}_0.$$

For $\mathbf{P}_0 = [\,1/2 \quad 1/2\,]^T$, $P_{10} = P_{01} = r$ and $P_{11} = P_{00} = 1 - r$, we obtain

$$\mathbf{P}_1 = \mathbf{P}\mathbf{P}_0 = [\,1/2 \quad 1/2\,]^T$$
$$\mathbf{P}_2 = \mathbf{P}\mathbf{P}_1 = [\,1/2 \quad 1/2\,]^T$$
$$\vdots$$
$$\mathbf{P}_n = \mathbf{P}\mathbf{P}_{n-1} = [\,1/2 \quad 1/2\,]^T.$$

5.17 *a*) We consider the conditional density

$$f = f_{X_n|X_{n-2}X_{n-3}\ldots X_1}(x_n|x_{n-2}, x_{n-3}, \ldots, x_1).$$

From the definition of conditional probability density, we have

$$f = \int_{-\infty}^{\infty} f_{X_n|X_{n-1}X_{n-2}\ldots X_1}(x_n|x_{n-1}, x_{n-2}, \ldots, x_1)$$
$$\times f_{X_{n-1}|X_{n-2}X_{n-3}\ldots X_1}(x_{n-1}|x_{n-2}, x_{n-3}, \ldots, x_1)dx_{n-1}.$$

If

$$f_{X_n|X_{n-1}X_{n-2}\ldots X_1} = f_{X_n|X_{n-1}} \tag{*}$$

for all n, then

$$f = \int_{-\infty}^{\infty} f_{X_n|X_{n-1}}(x_n|x_{n-1}) f_{X_{n-1}|X_{n-2}}(x_{n-1}|x_{n-2})dx_{n-1},$$

which is independent of $x_{n-3}, x_{n-4}, \ldots, x_1$. Therefore,

$$f_{X_n|X_{n-2}X_{n-3}\ldots X_1} = f_{X_n|X_{n-2}}.$$

By the same method it can be shown that (*) implies

$$f_{X_n|X_{n-m}X_{n-m-1}\ldots X_1} = f_{X_n|X_{n-m}}$$

for all $m \geq 0$. Furthermore, it follows from this that

$$f_{X_n|X_{n-m_1}X_{n-m_2}\ldots} = f_{X_n|X_{n-m_1}}$$

for all $m_1 \leq m_2 \leq m_3 \ldots$, since if $X_n|X_{n-m_1}$ is independent of $X_{n-m_1-1}, X_{n-m_1-2}, \ldots, X_1$, then it is certainly independent of a subset of these random variables. This final result can be reexpressed as

$$f_{X_{n_m}|X_{n_{m-1}}\ldots X_{n_1}} = f_{X_{n_m}|X_{n_{m-1}}} \tag{**}$$

for all $n_m \geq n_{m-1} \geq \ldots \geq n_1$. Thus, we have proved that (**) follows from (*). It can also be seen from this argument that if (*) is translation invariant, then so too is (**).

b) If the process of interest has starting time t_0 and known starting value $x_0 \neq 0$, then without loss of generality we can work with the process obtained by subtracting the constant x_0 and shifting in time by t_0 so that $X(0) = 0$. Since the events $(X(t_i) = x_i$ and $X(t_j) = x_j)$ and $(X(t_i) = x_i$ and $X(t_j) - X(t_i) = x_j - x_i)$ are identical, then their probability densities are equal. It follow that

$$f_{X(t_n)\ldots X(t_1)}(x_n, \ldots, x_1)$$
$$= f_{X(t_n)-X(t_{n-1}),\ldots,X(t_2)-X(t_1),X(t_1)}(x_n - x_{n-1}, \ldots, x_2 - x_1, x_1).$$

Now, if $X(0) = 0$, then $X(t_1) = X(t_1) - X(0)$, and if $X(t)$ has independent increments, then the above equation reduces to

$$f_{X(t_n)\ldots X(t_1)}(x_n, \ldots, x_1)$$
$$= f_{X(t_n)-X(t_{n-1})}(x_n - x_{n-1}) \cdots f_{X(t_2)-X(t_1)}(x_2 - x_1) f_{X(t_1)}(x_1).$$

By the same argument,

$$f_{X(t_i)-X(t_j)}(x_i - x_j) = \frac{f_{X(t_i)-X(t_j)}(x_i - x_j) f_{X(t_j)}(x_j)}{f_{X(t_j)}(x_j)} = \frac{f_{X(t_i)X(t_j)}(x_i, x_j)}{f_{X(t_j)}(x_j)}.$$

Consequently,

$$f_{X(t_n)\ldots X(t_1)}(x_n, \ldots, x_1) = \frac{f_{X(t_n)X(t_{n-1})}(x_n, x_{n-1}) \cdots f_{X(t_2)X(t_1)}(x_2, x_1)}{f_{X(t_{n-1})}(x_{n-1}) \cdots f_{X(t_1)}(x_1)}.$$

c) From the definition of conditional probability density, we have

$$f_{X(t_n)|X(t_{n-1})\ldots X(t_1)} = \frac{f_{X(t_n)\ldots X(t_1)}}{f_{X(t_{n-1})\ldots X(t_1)}}.$$

Using the result from part b in both the numerator and denominator yields

$$f_{X(t_n)|X(t_{n-1})\ldots X(t_1)} = \frac{f_{X(t_n)X(t_{n-1})}}{f_{X(t_{n-1})}} = f_{X(t_n)|X(t_{n-1})}.$$

Therefore, $X(t)$ is a Markov process.

5.18 *a*) If the first increments $\{Y_n - Y_{n-1}\}$ are independent for all n and are stationary, then since

$$Y_n - Y_m = (Y_n - Y_{n-1}) + (Y_{n-1} - Y_{n-2}) + \cdots + (Y_{m+1} - Y_m)$$

it follows that $Y_{n_1} - Y_{m_1}$ and $Y_{n_2} - Y_{m_2}$ are independent for all $n_1 \geq m_1 \geq n_2 \geq m_2$, and all such increments are stationary. Thus we shall consider first increments only.

The first increments of the binomial counting process

$$Y_n = \sum_{i=1}^{n} X_i$$

are given by

$$Y_n - Y_{n-1} = X_n.$$

Since $\{X_n\}$ are i.i.d., then the increments are independent and stationary. Since

$$Y_n = Y_{n-1} + X_n,$$

and $Y_{n-2}, Y_{n-3}, \ldots, Y_1$ are independent of X_n (since they depend only on $\{X_m, m \neq n\}$), then $(Y_n | Y_{n-1} = y_*) = X_n + y_*$ is independent of $Y_{n-2}, Y_{n-3}, \ldots, Y_1$ and, therefore, $\{Y_n\}$ is a Markov process (cf. exercise 5.17a). Moreover, it follows that the transition density is given by

$$f_{Y_n|Y_{n-1}}(y | y_*) = f_{X_n}(y - y_*),$$

which is a stationary probability density. Therefore, this Markov process is homogeneous (cf. exercise 5.17a).

Since the random walk process W_n is given in terms of the binomial counting process Y_n by

$$W_n = 2Y_n - n,$$

then the Markov property of Y_n obviously transfer to W_n; also the increments of W_n are given by

$$W_n - W_{n-1} = 2(Y_n - Y_{n-1}) - 1$$

and, therefore, the independence and stationarity properties of $Y_n - Y_{n-1}$ obviously carry over to $W_n - W_{n-1}$. Since

$$(W_n \,|W_{n-1}=w_*) = (2Y_n - n \,|Y_{n-1}=\frac{1}{2}[w_*+n-1])$$

$$= (2\{X_n + \frac{1}{2}[w_* + n - 1]\} - n) = 2X_n + w_* - 1$$

and $\{X_n\}$ is stationary, then the probability density $f_{W_n|W_{n-1}}$ is independent of n. Thus, this Markov process is homogeneous.

b) For the sampled and held process (4.29), $X(t_n)$ and $X(t_{n-1})$ are either independent or identical to each other; thus,

$$f_{X(t_n)|X(t_{n-1})}(x_n|x_{n-1}) = \begin{cases} f_{X(t_n)}(x_n) \\ \quad \text{or} \\ \delta(x_n - x_{n-1}). \end{cases}$$

In either case, being given $X(t_{n-2}), X(t_{n-3}), \ldots$, will not change this probability density. Therefore, $X(t)$ is a Markov process.

5.19 *a*) (i) Use of the Chapman-Kolmogorov equation yields

$$E\{E\{X(t_3)|X(t_2)\}|X(t_1)\} = \int_{-\infty}^{\infty} E\{X(t_3)|X(t_2)=x_2\} f_{X(t_2)|X(t_1)}(x_2|x_1)dx_2$$

$$= \iint_{-\infty}^{\infty} x_3 f_{X(t_3)|X(t_2)}(x_3|x_2)dx_3 f_{X(t_2)|X(t_1)}(x_2|x_1)dx_2$$

$$= \int_{-\infty}^{\infty} x_3 f_{X(t_3)|X(t_1)}(x_3|x_1)dx_3 = E\{X(t_3)|X(t_1)\}.$$

(ii) From the result of exercise 1.12 we have for a Gaussian process

$$E\{X(t_3) - m_X(t_3)|X(t_2)\} = \frac{K_X(t_3, t_2)}{K_X(t_2, t_2)}[X(t_2) - m_X(t_2)].$$

Therefore,

$$\frac{K_X(t_3, t_2)K_X(t_2, t_1)}{K_X(t_2, t_2)} = E\{\frac{K_X(t_3, t_2)}{K_X(t_2, t_2)}[X(t_2) - m_X(t_2)][X(t_1) - m_X(t_1)]\}$$

$$= E\{E\{X(t_3) - m_X(t_3)|X(t_2)\}[X(t_1) - m_X(t_1)]\}$$

$$= \iint_{-\infty}^{\infty} E\{X(t_3) - m_X(t_3)|X(t_2)=x_2\}[x_1 - m_X(t_1)] f_{X(t_2)X(t_1)}(x_2, x_1)dx_2dx_1$$

$$= \iiint_{-\infty}^{\infty} [x_3 - m_X(t_3)] f_{X(t_3)|X(t_2)}(x_3|x_2)dx_3[x_1 - m_X(t_1)] f_{X(t_2)X(t_1)}(x_2, x_1)dx_2dx_1$$

$$= \iiint_{-\infty}^{\infty} [x_3 - m_X(t_3)][x_1 - m_X(t_1)] f_{X(t_3)|X(t_2)X(t_1)}(x_3|x_2, x_1)$$

$$\times f_{X(t_2)X(t_1)}(x_2, x_1) dx_3 dx_2 dx_1$$

$$= \iint_{-\infty}^{\infty} [x_3 - m_X(t_3)][x_1 - m_X(t_1)] f_{X(t_3)X(t_1)}(x_1, x_3) dx_1 dx_3 = K_X(t_1, t_3).$$

(iii) From (5.22), we have

$$K_X(t_i, t_j) = \sigma^2 \exp\{-\alpha^2 |t_i - t_j|\}.$$

Therefore,

$$\frac{K_X(t_3, t_2) K_X(t_2, t_1)}{K_X(t_2, t_2)} = \frac{\sigma^4 \exp\{-\alpha^2[t_3 - t_2]\} \exp\{-\alpha^2[t_2 - t_1]\}}{\sigma^2}, \quad t_1 < t_2 < t_3$$

$$= \sigma^2 \exp\{-\alpha^2[(t_3 - t_2) + (t_2 - t_1)]\}$$

$$= \sigma^2 \exp\{-\alpha^2[t_3 - t_1]\} = K_X(t_3, t_1), \quad t_1 < t_2 < t_3.$$

b) Let $g(t) = e^{-t}$ and $h(t) = e^t$. Then, from (5.23), we have

$$K_X(t_1, t_2) = g[\max(t_1, t_2)] h[\min(t_1, t_2)] = g(t_2) h(t_1), \quad t_1 \le t_2$$
$$= \exp\{-t_2 + t_1\} = \exp\{-|t_1 - t_2|\}, \quad t_1 \le t_2.$$

Due to the symmetry of K_X, this result hold for $t_1 \ge t_2$ as well.

5.20 *a)* For a first-order Markov process, we have

$$f_{X_{n+p}|X_n X_{n-1}, \ldots, X_{n-q+1}} = f_{X_{n+p}|X_n}.$$

Therefore,

$$\hat{X}_{n+p} \triangleq E\{X_{n+p}|X_n, X_{n-1}, \ldots, X_{n-q+1}\} = E\{X_{n+p}|X_n\},$$

which depends on X_n only. For an m-th order Markov process, we have

$$f_{X_{n+p}|X_n X_{n-1}, \ldots, X_{n-q+1}} = f_{X_{n+p}|X_n X_{n-1}, \ldots, X_{n-m+1}}.$$

Therefore,

$$\hat{X}_{n+p} \triangleq E\{X_{n+p}|X_n, X_{n-1}, \ldots, X_{n-q+1}\}$$
$$= E\{X_{n+p}|X_n, X_{n-1}, \ldots, X_{n-m+1}\},$$

which depends on X_i, $i = n, n-1, \ldots, n-m+1$, only.

b) Let X_n be a Gaussian process with means and variances denoted by $m_{X_{n+p}} = E\{X_{n+p}\}$ and $\sigma^2_{X_{n+p}} = Var\{X_{n+p}\}$. Also, let $\mathbf{X} = [X_n \quad X_{n-1} \ldots X_{n-q+1}]^T$. Then, it can be shown (cf. exercise 1.12*d*) that the probability density function for X_{n+p} conditioned on $\mathbf{X}$ is Gaussian with mean and variance

$$m_{X_{n+p}|\mathbf{X}} = m_{X_{n+p}} + \mathbf{K}^T_{\mathbf{X}X_{n+p}}\mathbf{K}^{-1}_{\mathbf{XX}}(\mathbf{X}-\mathbf{m_X})$$
$$\sigma^2_{X_{n+p}|\mathbf{X}} = \sigma^2_{X_{n+p}} - \mathbf{K}^T_{\mathbf{X}X_{n+p}}\mathbf{K}^{-1}_{\mathbf{XX}}\mathbf{K}_{\mathbf{X}X_{n+p}},$$

where

$$\mathbf{K}_{\mathbf{X}X_{n+p}} = E\{\mathbf{X}X_{n+p}\}, \quad \mathbf{m_X} = E\{\mathbf{X}\}, \quad \mathbf{K_{XX}} = E\{(\mathbf{X}-\mathbf{m_X})(\mathbf{X}-\mathbf{m_X})^T\}.$$

Hence,

$$\hat{X}_{n+p} \triangleq E\{X_{n+p}|X_n, X_{n-1}, \ldots, X_{n-q+1}\} = m_{X_{n+p}} + \mathbf{K}^T_{\mathbf{X}X_{n+p}}\mathbf{K}^{-1}_{\mathbf{XX}}\,(\mathbf{X}-\mathbf{m_X}),$$

which is a constant plus a linear combination of the q available data points, in which the q–vector of coefficients is given by

$$\mathbf{c} = \mathbf{K}^T_{\mathbf{X}X_{n+p}}\mathbf{K}^{-1}_{\mathbf{XX}}$$

and the constant is given by

$$c = m_{X_{n+p}} - \mathbf{K}^T_{\mathbf{X}X_{n+p}}\mathbf{K}^{-1}_{\mathbf{XX}}\,\mathbf{m_X}.$$

5.21 Let $dX = X(t^+) - X(t)$ and $dt = t^+ - t$. Then, from (5.66c), we obtain

$$\begin{aligned} m(x,t)dt &= E\{dX(t)|X(t)=x\} = E\{X(t^+)-X(t)|X(t)=x\} \\ &= E\{X(t^+)|X(t)=x\} - E\{X(t)|X(t)=x\} = m_{X(t^+)|X(t)=x} - m_{X(t)|X(t)=x}. \end{aligned}$$

Thus,

$$m(x,t) = \frac{m_{X(t^+)|X(t)=x} - m_{X(t)|X(t)=x}}{dt} = \frac{\partial}{\partial s} m_{X(s)|X(t)=x}\Big|_{s=t^+}$$

Similarly, from (5.66d) we obtain

$$\begin{aligned} \sigma^2(x,t)dt &= E\{[dX(t)-m(x,t)dt]^2|X(t)=x\} \\ &= E\{[X(t^+)-m_{X(t^+)|X(t)=x} - (X(t)-m_{X(t)|X(t)=x})]^2|X(t)=x\} \\ &= E\{[X(t^+)-m_{X(t^+)|X(t)=x}]^2|X(t)=x\} \\ &\qquad - 2[x-m_{X(t)|X(t)=x}]E\{X(t^+) - m_{X(t^+)|X(t)=x}|X(t)=x\} \\ &\qquad + E\{[X(t)-m_{X(t)|X(t)=x}]^2|X(t)=x\} \\ &= \sigma^2_{X(t^+)|X(t)=x} - \sigma^2_{X(t)|X(t)=x}. \end{aligned}$$

Thus,

$$\sigma^2(x, t) = \frac{\sigma^2_{X(t^+)|X(t)=x} - \sigma^2_{X(t)|X(t)=x}}{dt} = \frac{\partial}{\partial s}\sigma^2_{X(s)|X(t)=x}\Big|_{s=t^+}.$$

5.22 (i) Substituting $m(x, t) = 0$ and $\sigma^2(x, t) = \alpha^2$ into (5.66a) yields the simplified Fokker-Planck equation

$$\frac{\partial}{\partial t}f(x, t) - \frac{\alpha^2}{2}\frac{\partial^2}{\partial x^2}f(x, t) = 0.$$

If

$$f(x, t) = \frac{1}{\sqrt{2\pi\alpha^2(t - t_0)}}\exp\{-\frac{1}{2}\frac{(x - x_0)^2}{\alpha^2(t - t_0)}\}, \qquad (*)$$

then

$$\frac{\partial}{\partial t}f(x, t) = -\frac{\pi\alpha^2}{[2\pi\alpha^2(t - t_0)]^{3/2}}\exp\{-\frac{1}{2}\frac{(x - x_0)^2}{\alpha^2(t - t_0)}\}$$
$$+ \frac{1}{\sqrt{2\pi\alpha^2(t - t_0)}}\frac{(x - x_0)^2}{2\alpha^2(t - t_0)^2}\exp\{-\frac{(x - x_0)^2}{\alpha^2(t - t_0)}\}$$

$$\frac{\partial^2}{\partial x^2}f(x, t) = -\frac{1}{\alpha^2(t - t_0)\sqrt{2\pi\alpha^2(t - t_0)}}\{-\frac{1}{2}\frac{(x - x_0)^2}{\alpha^2(t - t_0)}\}$$
$$+ \frac{(x - x_0)^2}{[\alpha^2(t - t_0)]^2\sqrt{2\pi\alpha^2(t - t_0)}}\exp\{-\frac{1}{2}\frac{(x - x_0)^2}{\alpha^2(t - t_0)}\}.$$

Thus, equation (*) satisfies the Fokker-Planck equation.

(ii) For a Markov process, we have, from (5.5),

$$f_{X(t_1)X(t_2)\cdots X(t_n)}(x_1, x_2, \ldots, x_n)$$
$$= f_{X(t_n)|X(t_{n-1})}(x_n|x_{n-1})f_{X(t_{n-1})|X(t_{n-2})}(x_{n-1}|x_{n-2})\cdots f_{X(t_1)}(x_1). \qquad (*)$$

Substituting

$$f_{X(t_j)|X(t_{j-1})}(x_j|x_{j-1}) = \frac{1}{\sqrt{2\pi\alpha^2(t_j - t_{j-1})}}\exp\{\frac{(x_j - x_{j-1})^2}{2\alpha^2(t_j - t_{j-1})}\}$$

into (*) yields (using $x_0 = t_0 = 0$)

$$f_{X(t_1)X(t_2)\cdots X(t_n)}(x_1, x_2, \ldots, x_n)$$
$$= \frac{1}{(2\pi\alpha^2)^{n/2}\prod_{j=1}^{n}(t_j - t_{j-1})^{1/2}}\exp\{\frac{1}{2\alpha^2}\sum_{j=1}^{n}\frac{(x_j - x_{j-1})^2}{t_j - t_{j-1}}\},$$

which is of the form of a joint Gaussian probability density (cf. (2.37)) with

$$|\mathbf{K_X}| = \prod_{j=1}^{n} \alpha^2(t_j - t_{j-1})$$

and

$$\mathbf{K_X^{-1}} = diag\left\{\frac{1}{\alpha^2(t_n - t_{n-1})}, \frac{1}{\alpha^2(t_{n-1} - t_{n-2})}, \ldots, \frac{1}{\alpha^2(t_1 - t_0)}\right\}.$$

5.23 (i) From (5.27), we obtain

$$\begin{aligned} 0 &= E\{X(t_n) - X(t_{n-1}) | X(t_1), X(t_2), \ldots, X(t_{n-1})\} \\ &= E\{X(t_n) | X(t_1), X(t_2), \ldots, X(t_{n-1})\} \\ &\quad - E\{X(t_{n-1}) | X(t_1), X(t_2), \ldots, X(t_{n-1})\} \\ &= E\{X(t_n) | X(t_1), X(t_2), \ldots, X(t_{n-1})\} - X(t_{n-1}). \end{aligned}$$

Therefore,

$$E\{X(t_n) | X(t_1), X(t_2), \ldots, X(t_{n-1})\} = X(t_{n-1}).$$

(ii) Since an independent-increment process with known initial value is a Markov process, then

$$f_{X(t_n)|X(t_1)X(t_2)\ldots X(t_{n-1})} = f_{X(t_n)|X(t_{n-1})}$$

and (5.28), therefore, reduces to (5.30).

5.24 *a*) Using the properties of conditional expectation in Chapter 2, we obtain

$$\begin{aligned} &E\{Y_n | Y_1, Y_2, \ldots, Y_{n-1}\} \\ &\quad = E\{E\{Z | X_1, \ldots, X_n\} | g(X_1, X_2, \ldots, X_{n-1})\} \\ &\quad = E_{X_n}\{E\{E\{(Z | X_n) | X_1, \ldots, X_{n-1}\} | g(X_1, \ldots, X_{n-1})\} | X_n\} \quad \text{(using (2.45))} \\ &\quad = E_{X_n}\{E\{(Z | X_n) | g(X_1, \ldots, X_{n-1})\}\} \quad \text{(using (2.47))} \\ &\quad = E\{Z | g(X_1, \ldots, X_{n-1})\} \quad \text{(using (2.45))} \\ &\quad = E\{E\{Z | X_1, \ldots, X_{n-1}\} | g(X_1, \ldots, X_{n-1})\} \quad \text{(using (2.47))} \\ &\quad = E\{Y_{n-1} | Y_1, Y_2, \ldots, Y_{n-1}\} = Y_{n-1}, \end{aligned}$$

where $E_{X_n}\{\cdot\}$ is the expected value with respect to X_n only. Hence, Y_n is a martingale process.

b) We have $Y(t) = X(t) - m_X(t)$, in which the initial value $X(t_0)$ is known (thus $m_X(t) = E\{X(t) | X(t_0)\}$). Let Z_i denote the independent increments of $X(t)$. Then

$$
\begin{aligned}
Z_n &= X(t_n) - X(t_{n-1}) = Y(t_n) - Y(t_{n-1}) + m_X(t_n) - m_X(t_{n-1}) \\
Z_{n-1} &= X(t_{n-1}) - X(t_{n-2}) = Y(t_{n-1}) - Y(t_{n-2}) + m_X(t_{n-1}) - m_X(t_{n-2}) \\
&\ \vdots \\
Z_1 &= X(t_1) - X(t_0) = Y(t_1) - Y(t_0) + m_X(t_1) - m_X(t_0)
\end{aligned}
$$

and

$$
\begin{aligned}
E\{Z_n - m_X(t_n) + m_X(t_{n-1}) | Z_1, Z_2, \ldots, Z_{n-1}\} &= E\{Z_n - m_X(t_n) + m_X(t_{n-1})\} \\
&= E\{Y(t_n) - Y(t_{n-1})\} = 0 - 0 = 0
\end{aligned}
$$

since $Y(t)$ has zero mean. Thus,

$$
E\{Y(t_n) - Y(t_{n-1}) | Y(t_0), Y(t_1), \ldots, Y(t_{n-1})\} = 0 \qquad (Y(t_0) \text{ is known})
$$

and, therefore,

$$
\begin{aligned}
&E\{Y(t_n) | Y(t_0), Y(t_1), \ldots, Y(t_{n-1})\} \\
&\qquad = E\{Y(t_{n-1}) | Y(t_0), Y(t_1), \ldots, Y(t_{n-1})\} = Y(t_{n-1}),
\end{aligned}
$$

which is the defining property of a martingale process.

5.25 *a*) Since $Z(t) = aX(t) + bY(t)$, then

$$
m_Z(t) = E\{Z(t)\} = aE\{X(t)\} + bE\{Y(t)\} = am_X(t) + bm_Y(t).
$$

Thus, $m_Z(t+u) = m_Z(t)$ for all a and b if and only if $m_X(t+u) = m_X(t)$ and $m_Y(t+u) = m_Y(t)$. Also,

$$
\begin{aligned}
R_Z(t_1, t_2) &= E\{Z(t_1)Z(t_2)\} \\
&= E\{a^2X(t_1)X(t_2) + abX(t_1)Y(t_2) + abX(t_2)Y(t_1) + b^2Y(t_1)Y(t_2)\} \\
&= a^2R_X(t_1, t_2) + abR_{XY}(t_1, t_2) + abR_{YX}(t_1, t_2) + b^2R_Y(t_1, t_2),
\end{aligned}
$$

and, therefore, $R_Z(t_1+u, t_2+u) = R_Z(t_1, t_2)$ for all a and b if and only if

$$
R_X(t_1+u, t_2+u) = R_X(t_1, t_2), \qquad R_Y(t_1+u, t_2+u) = R_Y(t_1, t_2),
$$

and

$$
R_{XY}(t_1+u, t_2+u) = R_{XY}(t_1, t_2).
$$

b) Let $X(t) = X_r(t) + X_i(t)$, where $X_r(t)$ and $X_i(t)$ are the real and imaginary parts of $X(t)$. Then

$$
X_r(t) = \frac{1}{2}X(t) + \frac{1}{2}X^*(t) \quad \text{and} \quad X_i(t) = \frac{1}{2}X(t) - \frac{1}{2}X^*(t)
$$

and, consequently,

$$R_{X_r}(t_1, t_2) = \frac{1}{4}R_X(t_1, t_2) + \frac{1}{4}R_X^*(t_1, t_2) + \frac{1}{4}R_{XX^*}(t_1, t_2) + \frac{1}{4}R_{XX^*}^*(t_1, t_2)$$
$$= \frac{1}{2}\mathrm{Re}\{R_X(t_1, t_2)\} + \frac{1}{2}\mathrm{Re}\{R_{XX^*}(t_1, t_2)\}$$
$$R_{X_i}(t_1, t_2) = \frac{1}{4}R_X(t_1, t_2) + \frac{1}{4}R_X^*(t_1, t_2) - \frac{1}{4}R_{XX^*}(t_1, t_2) - \frac{1}{4}R_{XX^*}^*(t_1, t_2)$$
$$= \frac{1}{2}\mathrm{Re}\{R_X(t_1, t_2)\} - \frac{1}{2}\mathrm{Re}\{R_{XX^*}(t_1, t_2)\}$$
$$R_{X_iX_r}(t_1, t_2) = \frac{1}{4}R_X(t_1, t_2) - \frac{1}{4}R_X^*(t_1, t_2) - \frac{1}{4}R_{XX^*}(t_1, t_2) + \frac{1}{4}R_{XX^*}^*(t_1, t_2)$$
$$= \frac{1}{2}\mathrm{Im}\{R_X(t_1, t_2)\} - \frac{1}{2}\mathrm{Im}\{R_{XX^*}(t_1, t_2)\}.$$

Hence, $X_r(t)$ and $X_i(t)$ are jointly WSS if and only if $R_X(t_1, t_2) = R_X(t_1 - t_2)$ and $R_{XX^*}(t_1, t_2) = R_{XX^*}(t_1 - t_2)$.

None of the three processes $A(t)\cos(\omega t)$, $A(t)\sin(\omega t)$, $A(t)e^{i\omega t}$ satisfy the above necessary and sufficient conditions.

5.26 Since $\{X_n\}$ are independent, then

$$f_{X_1X_2 \ldots X_n} = f_{X_1} f_{X_2} \cdots f_{X_n}.$$

For stationarity, we require

$$f_{X_{1+m}X_{2+m} \ldots X_{n+m}} = f_{X_1} f_{X_2} \cdots f_{X_n}$$

for all m and n. Thus, for this process, we require $f_{X_{p+m}} = f_{X_p}$ for all p and m, which is equivalent to requiring that f_{X_p} does not depend on p; that is, the random variables $\{X_n\}$ must be identically distributed. In this case, all order probability densities are completely specified by the single unique first-order probability density. Because of independence, we have

$$f_{X_n | X_{n-1}X_{n-2} \ldots X_{n-m}} = f_{X_n} = f_{X_n | X_{n-1}}.$$

Thus, this is a Markov-process.

5.27 *a*) We have

$$X_n = aX_{n-1} + bX_{n-2} + Z_n,$$

where Z_n is independent of $X_{n-1}, X_{n-2}, \ldots, X_1$. Thus,

$$f_{X_n | X_{n-1}X_{n-2} \cdots X_1}(x_n | x_{n-1}, x_{n-2}, \ldots, x_1) = f_{Z_n | X_{n-1}X_{n-2} \ldots X_1}(x_n - ax_{n-1} - bx_{n-2})$$
$$= f_{Z_n | X_{n-1}X_{n-2}}(x_n - ax_{n-1} - bx_{n-2}) = f_{X_n | X_{n-1}X_{n-2}}(x_n).$$

Hence, X_n is a second-order Markov process.

b) For $n \le n_0$, we have

$$E\{X_n\} = aE\{X_{n-1}\} + bE\{X_{n-2}\} + E\{Z_n\}$$

or

$$m_X = am_X + bm_X + m_Z.$$

Thus,

$$m_Z = m_X(1-a-b).$$

For $n = n_0 + 1$, we have

$$E\{X_{n_0+1}\} = aE\{X_{n_0}\} + bE\{X_{n_0-1}\} + E\{Z_{n_0+1}\} = am_X + bm_X + m_Z = m_X,$$

where the last equality follows from the result for $n \le n_0$. Thus,

$$m_X(n_0+1) = m_X.$$

Similarly,

$$m_X(n_0+i) = m_X \quad \text{for } i = 2, 3, 4, \ldots.$$

Furthermore, let $n = n_0 + 1$ and $k \le n_0$; then we have

$$X_{n_0+1} = aX_{n_0} + bX_{n_0-1} + Z_{n_0+1}$$

and

$$\begin{aligned} R_X(n_0+1, k) \triangleq E\{X_{n_0+1}X_k\} &= aE\{X_{n_0}X_k\} + bE\{X_{n_0-1}X_k\} + E\{Z_{n_0+1}X_k\} \\ &= aR_X(n_0-k) + bR_X(n_0-1-k) + m_Z m_X \\ &= aR_X(n_0+1-k-1) + bR_X(n_0+1-k-2) + m_Z m_X, \end{aligned}$$

which is a function of only $n_0 + 1 - k$. Therefore,

$$R_X(n_0+1, k) = R_X(n_0+1-k).$$

Similarly,

$$R_X(n_0+2, k) = R_X(n_0+2-k) \quad \text{for} \quad k \le n_0 + 1$$

and, likewise,

$$R_X(n_0+i, k) = R_X(n_0+i-k) \quad \text{for} \quad k \le n_0 + i - 1, \quad \text{for} \quad i = 3, 4, 5, \ldots.$$

Therefore, since $E\{X_n\} = m_X$ for $n > n_0$ and $E\{X_n X_m\}$ depends only on the difference $n - m$ for $n, m > n_0$, then $\{X_n\}$ is a WSS process for $n > n_0$.

5.28 We have $X_{n+1} = aX_n + Z_n$ for $-\infty < n < \infty$. This is the response of a first-order system (linear constant coefficient difference equation) to a stationary excitation. As long as this system is stable, its response should be stationary also. However, if the system is unstable, the response will grow without bound, in which case it would not be stationary. The system is stable if and only if $|a| < 1$.

5.29 *a*) Let $Y = v_1 X(t_2) + v_2 X(t_2) + \cdots + v_n X(t_n) = \mathbf{v}^T \mathbf{X}$, then

$$\mathbf{v}^T \mathbf{R} \mathbf{v} = \mathbf{v}^T E\{\mathbf{X}\mathbf{X}^T\}\mathbf{v} = E\{\mathbf{v}^T \mathbf{X}\mathbf{X}^T \mathbf{v}\} = E\{Y^2\} > 0.$$

b) Since $\mathbf{R} = E\{\mathbf{X}\mathbf{X}^T\}$, then

$$\mathbf{R}^T = [E\{\mathbf{X}\mathbf{X}^T\}]^T = E\{[\mathbf{X}\mathbf{X}^T]^T\} = E\{\mathbf{X}\mathbf{X}^T\} = \mathbf{R}.$$

c) The explicit formula for the correlation matrix is given by

$$\mathbf{R} = E\{\mathbf{X}\mathbf{X}^T\} = E\{\begin{bmatrix} X(t_1) \\ X(t_2) \\ \vdots \\ X(t_n) \end{bmatrix} \begin{bmatrix} X(t_1) & X(t_2) & \cdots & X(t_n) \end{bmatrix}\}$$

$$= \begin{bmatrix} E\{X(t_1)X(t_1)\} & E\{X(t_1)X(t_2)\} & \cdots & E\{X(t_1)X(t_n)\} \\ E\{X(t_2)X(t_1)\} & E\{X(t_2)X(t_2)\} & \cdots & E\{X(t_2)X(t_n)\} \\ \vdots & \vdots & & \vdots \\ E\{X(t_n)X(t_1)\} & E\{X(t_n)X(t_2)\} & \cdots & E\{X(t_n)X(t_n)\} \end{bmatrix}$$

$$= \begin{bmatrix} R_X(0) & R_X(t_1-t_2) & R_X(t_1-t_3) & \cdots & & R_X(t_1-t_n) \\ R_X(t_2-t_1) & R_X(0) & R_X(t_2-t_3) & \cdots & & R_X(t_2-t_n) \\ \vdots & \vdots & \vdots & & & \vdots \\ & & & & & R_X(t_{n-1}-t_n) \\ R_X(t_n-t_1) & R_X(t_n-t_2) & R_X(t_n-t_3) & \cdots & R_X(t_n-t_{n-1}) & R_X(0) \end{bmatrix}$$

Therefore, when $t_1 - t_2 = t_2 - t_3 = \cdots = t_{n-1} - t_n$, then $\mathbf{R}$ is Toeplitz.

5.30 *a*) Since $X(t)$ is WSS, then we have

(i)

$$R_X(\tau) = E\{X(t+\tau)X(t)\} = E\{X(t)X(t+\tau)\} = E\{X(t+\tau-\tau)X(t+\tau)\}$$
$$= E\{X(t'-\tau)X(t')\} = R_X(-\tau), \qquad t' = t + \tau.$$

(ii) Using the Cauchy-Schwarz inequality, we have

$$|R_X(\tau)| = |E\{X(t+\tau)X(t)\}| \le E\{X^2(t+\tau)\}^{1/2}E\{X^2(t)\}^{1/2} = R_X(0).$$

(iii) Using the Cauchy-Schwarz inequality, we have

$$|E\{X(t)[X(t+\tau+\varepsilon) - X(t+\tau)]\}| \le E\{X^2(t)\}^{1/2}E\{[X(t+\tau+\varepsilon) - X(t+\tau)]^2\}^{1/2},$$

which is equivalent to

$$|R_X(\tau+\varepsilon) - R_X(\tau)| \le [2R_X(0)\{R_X(0) - R_X(\varepsilon)\}]^{1/2},$$

which reveals that

$$\lim_{\varepsilon\to 0} R_X(\tau+\varepsilon) = R_X(\tau) \quad \text{if} \quad \lim_{\varepsilon\to 0} R_X(\varepsilon) = R_X(0).$$

Hence R_X is continuous at τ if it is continuous at 0.

b) The *MSE* is given by

$$\begin{aligned} MSE &\triangleq E\{[X(t) - X(t-\tau_0)]^2\} = E\{X^2(t) + X^2(t-\tau_0) - 2X(t)X(t-\tau_0)\} \\ &= 2[R_X(0) - R_X(\tau_0)]. \end{aligned}$$

Thus, if $R_X(\tau_0) = R_X(0)$, then $MSE = 0$ for all t.

5.31 Consider $Y(t) = X(t-\Theta)$, where Θ is uniformly distributed over $(-T/2, T/2]$ and is independent of $X(t)$, and assume that $m_X(t+T) = m_X(t)$ and $R_X(t_1+T, t_2+T) = R_X(t_1, t_2)$. Then the mean of $Y(t)$ is given by

$$\begin{aligned} m_Y(t) &= E\{E\{X(t-\Theta)|\Theta\}\} = E\{m_X(t-\Theta)\} \\ &= \frac{1}{T}\int_{-T/2}^{T/2} m_X(t-\theta)d\theta = \int_{t-T/2}^{t+T/2} m_X(u)du \\ &= \frac{1}{T}\int_{t-T/2}^{-T/2} m_X(u)du + \frac{1}{T}\int_{-T/2}^{T/2} m_X(u)du + \frac{1}{T}\int_{T/2}^{t+T/2} m_X(u)du \\ &= \frac{1}{T}\int_{-T/2}^{T/2} m_X(u)du = m_Y, \end{aligned}$$

since

$$\int_{t-T/2}^{-T/2} m_X(u)du = \int_{t+T/2}^{T/2} m_X(u)du = -\int_{T/2}^{t+T/2} m_X(u)du$$

(because $m_X(u+T) = m_X(u)$). Similarly, the autocorrelation of $Y(t)$ is given by

$$R_Y(t+\tau, t) = E\{E\{X(t+\tau-\Theta)X(t-\Theta)|\Theta\}\} = E\{R_X(t+\tau-\Theta, t-\Theta)\}$$

$$= \frac{1}{T}\int_{-T/2}^{T/2} R_X(t+\tau-\theta,\, t-\theta)d\theta = \frac{1}{T}\int_{t-T/2}^{t+T/2} R_X(u+\tau,\, u)du$$

$$= \frac{1}{T}\int_{t-T/2}^{-T/2} R_X(u+\tau,\, u)du + \frac{1}{T}\int_{-T/2}^{T/2} R_X(u+\tau,\, u)du + \frac{1}{T}\int_{T/2}^{t+T/2} R_X(u+\tau,\, u)du$$

$$= \frac{1}{T}\int_{-T/2}^{T/2} R_X(u+\tau,\, u)du = R_Y(\tau)$$

by the same reasoning used for m_Y. The autocovariance of $Y(t)$ is given by

$$K_Y(\tau) = R_Y(\tau) - m_Y^2 = \frac{1}{T}\int_{-T/2}^{T/2} R_X(u+\tau,\, u)du - \frac{1}{T^2}\iint_{-T/2}^{T/2} m_X(u)m_X(v)dudv.$$

This result does not in general reduce further unless $m_X(t)$ is constant, in which case we obtain

$$K_Y(\tau) = \frac{1}{T}\int_{-T/2}^{T/2} R_X(u+\tau,\, u)du - m_X^2 = \frac{1}{T}\int_{-T/2}^{T/2} [R_X(u+\tau,\, u) - m_X^2]du$$

$$= \frac{1}{T}\int_{-T/2}^{T/2} K_X(u+\tau,\, u)du.$$

5.32 From (4.29), we have

$$X(t) = \sum_{n=-\infty}^{\infty} V(nT)h(t-nT).$$

Therefore,

$$m_X(t) = E\{X(t)\} = \sum_{n=-\infty}^{\infty} E\{V(nT)\}h(t-nT)$$

$$= \sum_{n=-\infty}^{\infty} m_V h(t-nT) = \sum_{n=-\infty}^{\infty} m_V h(t-[n-1]T) = m_X(t+T).$$

Similarly, from (4.31), we have

$$R_X(t_1, t_2) = K_X(t_1, t_2) + m_X(t_1)m_X(t_2)$$

$$= \sum_{n=-\infty}^{\infty} \sigma_V^2 h(t_1-nT)h(t_2-nT) + m_X(t_1)m_X(t_2) = R_X(t_1+T,\, t_2+T).$$

Thus, $X(t)$ is cyclostationary in the wide sense with period T.
From (4.38), we have

$$m_Z(t) = m_Y E\{\sin(\omega_0 t + \Phi)\} = m_Z(t+T),$$

where $T = 2\pi/\omega_0$. From (4.39), we have

$$R_Z(t_1, t_2) = \frac{1}{2}R_Y(t_1-t_2)[\cos(\omega_0[t_1-t_2]) - E\{\cos(\omega_0[t_1+t_2]+2\Phi)\}]$$

$$= R_Z(t_1+T/2,\, t_2+T/2), \qquad T=2\pi/\omega_0.$$

Thus, $Z(t)$ is cyclostationary in the wide sense with period T. (If $m_Z(t) \equiv 0$, then $Z(t)$ is cyclostationary with period $T/2$.)

5.33 From exercise 4.15, we have

$$X(t) = \sum_{n=-\infty}^{\infty} p(t-nT-P_n), \qquad p(t) = \begin{cases} 1, & 0 \le t < \Delta \\ 0, & \textit{otherwise}, \end{cases}$$

where $\{P_n\}$ are i.i.d. Therefore,

$$m_X(t) = E\{X(t)\} = \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} p(t-nT-u)f_p(u)du$$

$$= \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} p(t-[n-1]T-u)f_p(u)du = m_X(t+T)$$

and, similarly,

$$R_X(t_1, t_2) = E\{X(t_1)X(t_2)\} = \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} p(t_1-nT-u)p(t_2-nT-u)f_p(u)du$$

$$+ \sum_{r\ne 0} \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} p(t_1-nT-u)f_p(u)du \int_{-\infty}^{\infty} p(t_2-nT-rT-v)f_p(v)dv$$

$$= R_X(t_1+T,\, t_2+T).$$

Thus, $X(t)$ is cyclostationary in wide sense with period T.

5.34 We have

$$|R_X(at, bt)| = |E\{X(at)X(bt)\}|$$

$$\le E\{X^2(at)\}^{1/2}E\{X^2(bt)\}^{1/2} \quad \to \quad 0 \text{ as } t \to \infty.$$

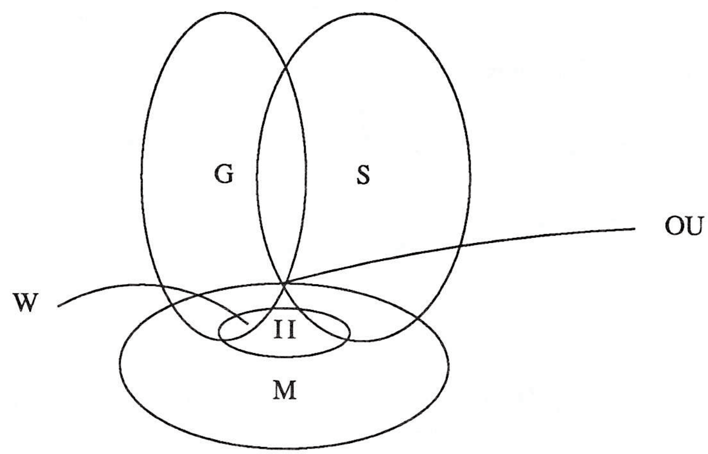
On the other hand, if $R_X(t_1, t_2) = R_X(t_1-t_2)$, then $R_X(at, bt) = R_X([a-b]t) = R_X(0)$ for all t when $a = b$, and therefore, $R_X(at, bt) \not\to 0$ as $t \to \infty$.

5.35 There is only one point in the intersection of G, S, and M.

S: Stationary process

G: Gaussian process

M: Markov process
W: Wiener process
OU: Ornstein-Uhlenbeck process
II: Independent-increment process.



Chapter 6

The Wiener and Poisson Processes

6.1 From (6.10) (or (4.23)), the variance of the process $W(t)$ in (6.6) is given by

$$Var\{W(t)\} = \lim_{\Delta_t \to 0} K_{W_\Delta}(t, t) = \lim_{\Delta_t \to 0} \Delta_w^2 \min\{n, n\} = \lim_{\Delta_t \to 0} \Delta_w^2 n.$$

Since $n = [t]/\Delta_t$ and $\Delta_w = \alpha \Delta_t^x$, then

$$Var\{W(t)\} = \lim_{\Delta_t \to 0} \frac{\Delta_w^2}{\Delta_t}[t] = \lim_{\Delta_t \to 0} \alpha^2 \Delta_t^{2x-1}[t] = \begin{cases} 0, & x > 1/2 \\ \alpha^2 t, & x = 1/2 \\ \infty, & x < 1/2. \end{cases}$$

Thus, $x = 1/2$ is the only viable value and this yields the Wiener process.

6.2 Let $0 \le t_1 \le t_2$. From (6.12) and (6.13), we obtain for the given process $X(t)$

$$m_X(t) = E\{X(t)\} = e^{-t} E\{W(e^{2t})\} = 0$$

$$\begin{aligned} K_X(t_1, t_2)\} &= E\{X(t_1)X(t_2)\} = e^{-t_1 - t_2} E\{W(e^{2t_1})W(e^{2t_2})\} \\ &= e^{-t_1 - t_2} \min\{e^{2t_1}, e^{2t_2}\} = e^{t_1 - t_2}. \end{aligned}$$

Therefore, $X(t)$ is WSS. Furthermore, since $W(t)$ is a Gauss-Markov process, then $X(t) = e^{-t} W(e^{2t})$ is also a Gauss-Markov process since neither amplitude scaling nor monotonic time warping destroys the Gaussian property or the Markov property. As a check, we see that

$$\frac{K_X(t_3, t_2) K_X(t_2, t_1)}{K_X(t_2, t_2)} = \frac{e^{t_2 - t_3} e^{t_1 - t_2}}{1} = e^{t_1 - t_3} = K_X(t_1, t_3),$$

which is (5.21); therefore, we see that if $X(t)$ is Gaussian, then it is a Markov process. In conclusion, since $X(t)$ is a stationary Gauss-Markov process, then it is the Ornstein-Uhlenbeck process.

6.3 Let $t_1 < t_2 < t_3 < t_4$; then by using (6.13) we obtain for the Wiener process $W(t)$

$$\begin{aligned} &E\{[W(t_1) - W(t_2)][W(t_3) - W(t_4)]\} \\ &= E\{W(t_1)W(t_3)\} - E\{W(t_1)W(t_4)\} - E\{W(t_2)W(t_3)\} + E\{W(t_2)W(t_4)\} \\ &= \alpha^2 t_1 - \alpha^2 t_1 - \alpha^2 t_2 + \alpha^2 t_2 = 0. \end{aligned}$$

Thus, the increments, which have zero mean values, are uncorrelated.

6.4 From (6.13), we have

$$R_W(t_1, t_2) = \alpha^2 \min\{t_1, t_2\} = \alpha^2[t_1 u(t_2-t_1) + t_2 u(t_1-t_2)].$$

Therefore, (6.19) yields

$$R_Z(t_1, t_2) = \frac{\partial^2}{\partial t_1 \partial t_2} R_W(t_1, t_2) = \frac{\partial}{\partial t_2}\alpha^2[u(t_2-t_1) - t_1\delta(t_2-t_1) + t_2\delta(t_1-t_2)]$$

$$= \frac{\partial}{\partial t_2}\alpha^2[u(t_2-t_1) + (t_2-t_1)\delta(t_2-t_1)]$$

$$= \frac{\partial}{\partial t_2}\alpha^2 u(t_2-t_1) = \alpha^2\delta(t_2-t_1) = \alpha^2\delta(t_1-t_2),$$

which is (6.20), as desired.

6.5 Using the result (6.20) yields

$$R_Y(t_1, t_2) = E\{Y(t_1)Y(t_2)\} = E\{X(t_1)X(t_2)\}E\{Z(t_1)Z(t_2)\}$$

$$= R_X(t_1, t_2)R_Z(t_1, t_2) = \alpha^2 R_X(t_1, t_2)\delta(t_1-t_2)$$

$$= \alpha^2 R_X(t_1, t_1)\delta(t_1-t_2) = \alpha^2[\sigma_X^2(t_1)+m_X^2(t_1)]\delta(t_1-t_2),$$

which is the desired result.

6.6 *a)* Substituting $w = r\Delta_w$ and $t = n\Delta_t$ into (6.52) yields

$$P_{W_\Delta} \simeq \frac{1}{\sqrt{2\pi t/\Delta_t}}\exp\{-\frac{1}{2}\frac{(w/\Delta_w)^2}{t/\Delta_t}\}.$$

Then, using the hint and (6.5) yields the following first-order probability density for the Wiener process $W(t)$:

$$f_{W(t)}(w) = \lim_{\Delta_w \to 0}\frac{P_{W_\Delta}}{\Delta_w}$$

$$= \lim_{\Delta_w \to 0}\frac{1}{\Delta_w\sqrt{2\pi\alpha^2 t/\Delta_w^2}}\exp\{-\frac{1}{2}\frac{(w/\Delta_w)^2}{\alpha^2 t/\Delta_w^2}\} = \frac{1}{\sqrt{2\pi\alpha^2 t}}\exp\{-\frac{1}{2}\frac{w^2}{\alpha^2 t}\}.$$

Since $\Delta_w = \alpha\sqrt{\Delta_t}$, then

$$\frac{w}{r} = \Delta_w = \alpha\sqrt{\Delta_t} = \alpha\left(\frac{t}{n}\right)^{1/2}$$

and, therefore, $r/\sqrt{n} = w/\alpha\sqrt{t}$. Consequently, we have the limit

$$\lim_{n\to\infty}\frac{r}{\sqrt{n}} = \lim_{n\to\infty}\frac{w}{\alpha\sqrt{t}} = \frac{w}{\alpha\sqrt{t}}$$

for fixed w and t. Thus, r is indeed on the order of $\sqrt{n}$ as $n \to \infty$.

b) Let $Z_i = W(t_i) - W(t_{i-1})$, $i = 1, 2, \ldots, n$; then $\{Z_i\}$ is an independent identically distributed sequence with Gaussian probability density. Thus, $\{Z_i\}$ are jointly Gaussian random variables and, therefore, for arbitrary $\{a_i\}$,

$$Y = \sum_{i=1}^{n} a_i Z_i = \sum_{i=1}^{n} a_i [W(t_i) - W(t_{i-1})]$$

is a Gaussian random variable. But Y can also be expressed as

$$Y = \sum_{i=0}^{n} b_i W(t_i),$$

where $b_0 = a_1$, $b_n = a_n$, and $b_i = a_i - a_{i+1}$, $i = 1, 2, \ldots, n-1$. If we let $t_0 = 0$, then $W(t_0) = 0$, and also, since $\{a_i\}_1^n$ are arbitrary, $\{b_i\}_1^n$ are arbitrary. Therefore, since Y is a Gaussian random variable, then the Wiener process $W(t)$ is a Gaussian random process.

6.7 *a*) Let $t_1 < t_2$; then, from (6.12) and (6.13), we have for the Wiener process $W(t)$

$$E\{W(t_1)\} = 0, \qquad Var\{W(t_1)\} = \alpha^2 t_1$$

$$E\{W(t_2)\} = 0, \qquad Var\{W(t_2)\} = \alpha^2 t_2$$

$$\gamma = \frac{K_W(t_1, t_2)}{\sqrt{Var\{W(t_1)\} Var\{W(t_2)\}}} = \frac{\alpha^2 t_1}{\alpha^2 \sqrt{t_1 t_2}} = \left[\frac{t_1}{t_2}\right]^{1/2},$$

from which it follows that $W(t)$ is nonstationary. However, using the result of exercise 1.12*d* for jointly Gaussian random variables yields

$$f_{W(t_2)|W(t_1)}(w_2|w_1) = \frac{1}{\sqrt{2\pi\alpha^2 t_2 (1-\gamma^2)}} \exp\{-\frac{1}{2} \frac{(w_2 - \gamma w_1 \sqrt{\alpha^2 t_2} / \sqrt{\alpha^2 t_1}\,)^2}{\alpha^2 t_2 (1-\gamma^2)}\}$$

$$= \frac{1}{\sqrt{2\pi\alpha^2 (t_2 - t_1)}} \exp\{-\frac{1}{2} \frac{(w_2 - w_1)^2}{\alpha^2 (t_2 - t_1)}\}$$

$$= f_{W(t_2+t)|W(t_1+t)}(w_2|w_1),$$

from which it follows that $W(t)$ is a homogeneous process. It follows from exercise 5.17*b* that $W(t)$ is a Markov process.

b) It is shown in part *a* that $f_{W(t)|W(t_0)}(w\,|w_0)$ is given by (6.53) with replacement of t with $t - t_0$ and w with $w - w_0$ for $t_0 \geq 0$:

$$f_{W(t)|W(t_0)}(w\,|w_0) = \frac{1}{\sqrt{2\pi\alpha^2 (t - t_0)}} \exp\{-\frac{1}{2} \frac{(w - w_0)^2}{\alpha^2 (t - t_0)}\}.$$

To see that this function satisfies the diffusion equation, we observe that

$$\frac{\partial}{\partial t} f_{W(t)|W(t_0)}(w\,|w_0) = \frac{1}{\sqrt{2\pi\alpha^2(t-t_0)}}\left[\frac{-1}{2(t-t_0)} + \frac{(w-w_0)^2}{2\alpha^2(t-t_0)^2}\right] \times \exp\left\{-\frac{1}{2}\frac{(w-w_0)^2}{\alpha^2(t-t_0)}\right\}$$

and

$$\frac{\partial^2}{\partial w^2} f_{W(t)|W(t_0)}(w\,|w_0) = \frac{1}{\sqrt{2\pi\alpha^2(t-t_0)}}\left[\frac{-1}{\alpha^2(t-t_0)} + \frac{(w-w_0)^2}{\alpha^4(t-t_0)^2}\right] \times \exp\left\{-\frac{1}{2}\frac{(w-w_0)^2}{\alpha^2(t-t_0)}\right\}.$$

Therefore, we have

$$\frac{\partial}{\partial t} f_{W(t)|W(t_0)}(w\,|w_0) = \frac{\alpha^2}{2}\frac{\partial^2}{\partial w^2} f_{W(t)|W(t_0)}(w\,|w_0).$$

To show that the diffusion equation follows from the Fokker-Planck equation, we observe that (5.66e) and (5.66f) yield

$$m(w,t) = \frac{\partial}{\partial s} m_{W(s)|W(t)=w}\Big|_{s=t^+} = \frac{\partial}{\partial s} w = 0$$

$$\sigma^2(w,t) = \frac{\partial}{\partial s}\sigma^2_{W(s)|W(t)=w}\Big|_{s=t^+} = \frac{\partial}{\partial s}\alpha^2(s-t) = \alpha^2.$$

Therefore, the Fokker-Planck equation (5.66a) reduces to

$$\frac{\partial}{\partial t} f_{W(t)|W(t_0)}(w\,|w_0) - \frac{1}{2}\frac{\partial^2}{\partial w^2}[\alpha^2 f_{W(t)|W(t_0)}(w\,|w_0)] = 0,$$

which is the diffusion equation.

6.8 Using the event definition $A \triangleq \{W(\cdot): W(s) = w \text{ for some } s \le t\}$ and the fact that $\{A, W(t) \ge w\} \equiv \{W(t) \ge w\}$ (since $W(0) = 0$) in the hint yields

$$P_W(t) \equiv P(A) = P(A, W(t) \le w) + P(W(t) \ge w).$$

Applying the definition of conditional probability density yields

$$P(A) = P(W(t) \le w\,|A)P(A) + P(W(t) \ge w).$$

Using the reflection principle (6.54) then yields

$$P(A) = \frac{1}{2}P(A) + P(W(t) \ge w),$$

from which we obtain

$$P_W(t) = 2P(W(t) \geq w) = 2[1 - F_{W(t)}(w)].$$

Since $m_{W(t)} = 0$ and $\sigma_{W(t)} \to \infty$ as $t \to \infty$, then $F_{W(t)} \to 1/2$ as $t \to \infty$. As a result, $P_W(t) \to 1$ as $t \to \infty$.

6.9 For the Poisson counting process $N(t)$, using (6.34) yields

$$E\{N(t)\} = \sum_{n=1}^{\infty} nP_t(n) = \sum_{n=1}^{\infty} ne^{-\lambda t}\frac{(\lambda t)^n}{n!} = e^{-\lambda t}\sum_{n=1}^{\infty}\frac{(\lambda t)^n}{(n-1)!} = \lambda t e^{-\lambda t}\sum_{m=0}^{\infty}\frac{(\lambda t)^m}{m!} = \lambda t.$$

Similarly,

$$E\{N(t+T)\} = \lambda(t+T).$$

Hence,

$$E\{N(t+T) - N(t)\} = E\{N(t+T)\} - E\{N(t)\} = \lambda(t+T) - \lambda t = \lambda T.$$

6.10 Let $t_1 < t_2$ and $N(t_1) = n_1 \leq N(t_2) = n_2$. Then, for the weighted Poisson counting process $W(t)$, we have

$$\begin{aligned}
R_W(t_1, t_2) &= E\{W(t_1)W(t_2)\} = E\{E\{W(t_1)W(t_2)|N(t_1) \text{ and } N(t_2)\}\} \\
&= E\{E\{\sum_{i=1}^{N(t_1)} Y_i \sum_{j=1}^{N(t_2)} Y_j |N(t_1) \text{ and } N(t_2)\}\} \\
&= E\{E\{\sum_{i=1}^{N(t_1)} Y_i \sum_{j=1}^{N(t_1)} Y_j + \sum_{i=1}^{N(t_1)} Y_i \sum_{j=N(t_1)+1}^{N(t_2)} Y_j |N(t_1) \text{ and } N(t_2)\}\} \\
&= E\{\sum_{i=1}^{N(t_1)} E\{Y_i^2\} + \sum_{\substack{i,j=1\\ i\neq j}}^{N(t_1)} E\{Y_iY_j\} + E\{\sum_{i=1}^{N(t_1)} Y_i\}E\{\sum_{j=N(t_1)+1}^{N(t_2)} Y_j\}|N(t_1) \text{ and } N(t_2)\} \\
&= E\{(\sigma_Y^2 + m_Y^2)N(t_1) + m_Y^2N(t_1)[N(t_1) - 1] + m_Y^2N(t_1)[N(t_2) - N(t_1)]\} \\
&= (\sigma_Y^2 + m_Y^2)E\{N(t_1)\} + m_Y^2E\{N^2(t_1) - N(t_1)\} + m_Y^2E\{N(t_1)\}E\{N(t_2) - N(t_1)\},
\end{aligned}$$

where the independence of Y_i and Y_j, for $i \neq j$, and of $N(t_1)$ and $N(t_2) - N(t_1)$ has been used. Carrying out the expectation in the above equation yields (with the use of $E\{N^2(t_1)\} = \lambda t_1 + (\lambda t_1)^2$)

$$\begin{aligned}
R_W(t_1, t_2) &= (\sigma_Y^2 + m_Y^2)\lambda t_1 + m_Y^2(\lambda t_1)^2 + m_Y^2\lambda t_1(\lambda t_2 - \lambda t_1) \\
&= (\sigma_Y^2 + m_Y^2)\lambda t_1 + m_Y^2(\lambda t_1)(\lambda t_2).
\end{aligned}$$

Hence, using (6.38) yields

$$K_W(t_1, t_2) = R_W(t_1, t_2) - m_W(t_1)m_W(t_2) = \lambda(\sigma_Y^2 + m_Y^2)t_1, \quad t_1 < t_2.$$

Similarly, it can be shown that

$$K_W(t_1, t_2) = \lambda(\sigma_Y^2 + m_Y^2)t_2, \quad t_2 < t_1.$$

Thus,

$$K_W(t_1, t_2) = \lambda(\sigma_Y^2 + m_Y^2)\min\{t_1, t_2\}.$$

Finally, to verify that $E\{N^2(t)\} = \lambda t + (\lambda t)^2$, we proceed as follows:

$$E\{N^2(t)\} = \sum_{n=1}^{\infty} n^2 P_t(n) = \sum_{n=1}^{\infty} n^2 \frac{(\lambda t)^n e^{-\lambda t}}{n!} = e^{-\lambda t} \sum_{n=1}^{\infty} \frac{[n(n-1)+n](\lambda t)^n}{n!}$$

$$= e^{-\lambda t}\left[(\lambda t)^2 \sum_{m=0}^{\infty} \frac{(\lambda t)^m}{m!} + \lambda t \sum_{q=0}^{\infty} \frac{(\lambda t)^q}{q!}\right] = (\lambda t)^2 + \lambda t.$$

6.11 From the definition of conditional probability density and (6.35) and (6.25), we obtain

$$Prob\{N(t_m)=n_m | N(t_{m-1})=n_{m-1} \ldots N(t_1)=n_1\}$$

$$= \frac{Prob\{N(t_1)=n_1, N(t_2)=n_2, \ldots, N(t_m)=n_m\}}{Prob\{N(t_1)=n_1, N(t_2)=n_2, \ldots, N(t_{m-1})=n_{m-1}\}}$$

$$= P_{t_m - t_{m-1}}(n_m - n_{m-1})$$

$$= \frac{Prob\{N(t_m)=n_m, N(t_{m-1})=n_{m-1}\}}{Prob\{N(t_{m-1})=n_{m-1}\}}$$

$$= Prob\{N(t_m)=n_m | N(t_{m-1})=n_{m-1}\}.$$

Therefore, by definition (cf.(5.11)), the Poisson counting process $N(t)$ is a Markov process. Furthermore, the preceding shows that the transition probability is time-translation invariant; therefore, this is a homogeneous Markov process.

6.12 *a*) For the asynchronous telegraph signal $Y(t)$, we have

$$E\{Y(t)\} = E\{(-1)^{N(t)}\} = \sum_{n=0}^{\infty} (-1)^n P_t(n)$$

$$= \sum_{\substack{n=0 \\ n=even}}^{\infty} P_t(n) - \sum_{\substack{n=1 \\ n=odd}}^{\infty} P_t(n) = \sum_{\substack{n=0 \\ n=even}}^{\infty} \frac{(\lambda t)^n e^{-\lambda t}}{n!} - \sum_{\substack{n=1 \\ n=odd}}^{\infty} \frac{(\lambda t)^n e^{-\lambda t}}{n!}$$

$$= e^{-\lambda t}\left\{\frac{1}{2}\sum_{n=0}^{\infty}\left[\frac{(\lambda t)^n}{n!} + \frac{(-\lambda t)^n}{n!}\right] - \frac{1}{2}\sum_{n=0}^{\infty}\left[\frac{(\lambda t)^n}{n!} - \frac{(-\lambda t)^n}{n!}\right]\right\}$$

$$= e^{-\lambda t}[\frac{e^{\lambda t} + e^{-\lambda t}}{2} - \frac{e^{\lambda t} - e^{-\lambda t}}{2}] = e^{-2\lambda t}.$$

b) Since

$$Y(t+\tau)Y(t) = (-1)^{N(t+\tau)+N(t)} = (-1)^{2N(t)+N(t+\tau)-N(t)}$$
$$= [(-1)^2]^{N(t)}(-1)^{N(t+\tau)-N(t)} = (-1)^N,$$

where $N = N(t+\tau) - N(t)$, then the autocorrelation for $Y(t)$ is given by

$$R_Y(\tau) = E\{(-1)^N\} = \sum_{\substack{n=0\\n=even}}^{\infty} P_{|\tau|}(n) - \sum_{\substack{n=1\\n=odd}}^{\infty} P_{|\tau|}(n)$$
$$= \sum_{\substack{n=0\\n=even}}^{\infty} \frac{(\lambda|\tau|)^n e^{-\lambda|\tau|}}{n!} - \sum_{\substack{n=1\\n=odd}}^{\infty} \frac{(\lambda|\tau|)^n e^{-\lambda|\tau|}}{n!}$$
$$= e^{-\lambda|\tau|}[\frac{e^{\lambda|\tau|} + e^{-\lambda|\tau|}}{2} - \frac{e^{\lambda|\tau|} - e^{-\lambda|\tau|}}{2}] = e^{-2\lambda|\tau|}.$$

c) Let $U(t) = AY(t)$. Then we have

$$m_U(t) = E\{U(t)\} = E\{A\}E\{Y(t)\} = (1\times\frac{1}{2} - 1\times\frac{1}{2})e^{-2\lambda t} = 0, \qquad t \ge 0$$
$$R_U(t_1, t_2) = E\{U(t_1)U(t_2)\} = E\{A^2\}E\{Y(t_1)Y(t_2)\}$$
$$= [1^2\times\frac{1}{2} + (-1)^2\times\frac{1}{2}]e^{-2\lambda|t_2-t_1|} = e^{-2\lambda|t_2-t_1|}, \qquad t_1, t_2 \ge 0.$$

Therefore, $U(t)$ is WSS for $t \ge 0$.

6.13 Let the time interval be T. Then

$$P[k|\lambda] = Prob\{N(t+T)-N(t)=k|\lambda\} = P_T(k|\lambda) = e^{-\lambda T}\frac{(\lambda T)^k}{k!},$$

where λ is the light intensity. Hence,

$$P[\lambda_1|k]P(k) = P[k|\lambda_1]P(\lambda_1) = e^{-\lambda_1 T}\frac{(\lambda_1 T)^k}{k!}P(\lambda_1)$$
$$P[\lambda_2|k]P(k) = P[k|\lambda_2]P(\lambda_2) = e^{-\lambda_2 T}\frac{(\lambda_2 T)^k}{k!}P(\lambda_2),$$

and we decide λ_1 if

$$\frac{P[\lambda_1|k]}{P[\lambda_2|k]} = \frac{P[\lambda_1|k]P(k)}{P[\lambda_2|k]P(k)} = e^{(\lambda_2-\lambda_1)T}\left[\frac{\lambda_1}{\lambda_2}\right]^k \frac{P(\lambda_1)}{P(\lambda_2)} > 1,$$

which is equivalent to

$$k > \frac{(\lambda_1-\lambda_2)T - \ln[P(\lambda_1)/P(\lambda_2)]}{\ln(\lambda_1/\lambda_2)}.$$

6.14 For the approximating process $\hat{X}(t)$, we have

$$m_{\hat{X}} = E\{\hat{X}(t)\} = \sum_{n=-\infty}^{\infty} E\{X_n\}h(t-n\Delta_t) = \sum_{n=-\infty}^{\infty} \lambda\Delta_t h(t-n\Delta_t).$$

Therefore,

$$m_X = \lim_{\Delta_t\to 0} m_{\hat{X}} = \lim_{\Delta_t\to 0} \lambda \sum_{n=-\infty}^{\infty} h(t-n\Delta_t)\Delta_t = \lambda\int_{-\infty}^{\infty} h(t-u)du = \lambda\int_{-\infty}^{\infty} h(u)du.$$

Also, since

$$R_{\hat{X}}(\tau) = E\{\hat{X}(t+\tau)\hat{X}(t)\} = \sum_{n,m=-\infty}^{\infty} E\{X_nX_m\}h(t+\tau-n\Delta_t)h(t-m\Delta_t)$$

$$= \sum_{n=-\infty}^{\infty} E\{X_n^2\}h(t+\tau-n\Delta_t)h(t-n\Delta_t)$$

$$+ \sum_{\substack{n,m=-\infty\\ n\neq m}}^{\infty} E\{X_n\}E\{X_m\}h(t+\tau-n\Delta_t)h(t-m\Delta_t)$$

$$= \sum_{n=-\infty}^{\infty} \lambda\Delta_t h(t+\tau-n\Delta_t)h(t-n\Delta_t) + \sum_{n,m=-\infty}^{\infty} \lambda^2\Delta_t^2 h(t+\tau-n\Delta_t)h(t-m\Delta_t)$$

$$- \sum_{n=-\infty}^{\infty} \lambda^2\Delta_t^2 h(t+\tau-n\Delta_t)h(t-n\Delta_t),$$

then

$$R_X(\tau) = \lim_{\Delta_t\to 0} R_{\hat{X}}(\tau)$$

$$= \lambda\int_{-\infty}^{\infty} h(t+\tau-u)h(t-u)du + \lambda^2\int_{-\infty}^{\infty} h(t+\tau-u)du\int_{-\infty}^{\infty} h(t-v)dv - 0$$

$$= \lambda\int_{-\infty}^{\infty} h(\tau+u)h(u)du + [\lambda\int_{-\infty}^{\infty} h(u)du]^2.$$

Therefore,

$$K_X(\tau) = R_X(\tau) - m_X^2 = \lambda\int_{-\infty}^{\infty} h(\tau+u)h(u)du.$$

6.15 *a*) Assume that $\{Y_i\}$ are independent and identically distributed. Similar to the approach in the preceding exercise, we can approximate the process by

$$\hat{X}(t) = \sum_{n=-\infty}^{\infty} Y_n X_n h(t - n\Delta_t),$$

where X_n is defined as before. Then,

$$m_{\hat{X}} = E\{\hat{X}(t)\} = \sum_{n=-\infty}^{\infty} E\{Y_n\}E\{X_n\}h(t-\Delta_t) = m_Y \sum_{n=-\infty}^{\infty} \lambda\Delta_t h(t - n\Delta_t)$$

and, therefore,

$$m_X = \lim_{\Delta_t \to 0} m_{\hat{X}} = \lambda m_Y \int_{-\infty}^{\infty} h(u)du.$$

Since

$$R_{\hat{X}}(\tau) = E\{\hat{X}(t+\tau)\hat{X}(t)\} = \sum_{n,m=-\infty}^{\infty} E\{Y_n Y_m\}E\{X_n X_m\}h(t+\tau-n\Delta_t)h(t-m\Delta_t)$$

$$= \sum_{n=-\infty}^{\infty} E\{Y_n^2\}E\{X_n^2\}h(t+\tau-n\Delta_t)h(t-n\Delta_t)$$

$$+ \sum_{\substack{n,m=-\infty \\ n\neq m}}^{\infty} E\{Y_n\}E\{Y_m\}E\{X_n\}E\{X_m\}h(t+\tau-n\Delta_t)h(t-m\Delta_t)$$

$$= (\sigma_Y^2 + m_Y^2) \sum_{n=-\infty}^{\infty} \lambda\Delta_t h(t+\tau-n\Delta_t)h(t-n\Delta_t)$$

$$+ m_Y^2 \sum_{n,m=-\infty}^{\infty} \lambda^2\Delta_t^2 h(t+\tau-n\Delta_t)h(t-m\Delta_t)$$

$$- m_Y^2 \sum_{n=-\infty}^{\infty} \lambda^2\Delta_t^2 h(t+\tau-n\Delta_t)h(t-n\Delta_t),$$

then

$$R_X(\tau) = \lim_{\Delta_t \to 0} R_{\hat{X}}(\tau) = \lambda(\sigma_Y^2 + m_Y^2) \int_{-\infty}^{\infty} h(\tau+u)h(u)du + \left[m_Y \lambda \int_{-\infty}^{\infty} h(u)du\right]^2.$$

Thus,

$$K_X(\tau) = R_X(\tau) - m_X^2 = \lambda(\sigma_Y^2 + m_Y^2) \int_{-\infty}^{\infty} h(\tau+u)h(u)du.$$

b) Again, if $\{Y_i\}$ are i.i.d., then we can approximate

$$X(t) = \sum_{i=1}^{N(t)} h(t - T_i, Y_i), \qquad t \geq 0$$

by

$$\hat{X}(t) = \sum_{n=-\infty}^{\infty} X_n h(t - n\Delta_t, Y_n)$$

with X_n defined as before. Then,

$$m_{\hat{X}} = E\{\hat{X}(t)\} = \sum_{n=-\infty}^{\infty} E\{X_n\}E\{h(t - n\Delta_t, Y_n)\} = \sum_{n=-\infty}^{\infty} \lambda\Delta_t E\{h(t - n\Delta_t, Y_n)\}$$

and

$$m_X = \lim_{\Delta_t \to 0} m_{\hat{X}} = \lambda \iint_{-\infty}^{\infty} h(u, y) f_Y(y) dy du.$$

Similarly, we obtain

$$R_{\hat{X}} = E\{\hat{X}(t+\tau)\hat{X}(t)\} = \sum_{n,m=-\infty}^{\infty} E\{X_n X_m\}E\{h(t+\tau-n\Delta_t, Y_n)h(t - m\Delta_t, Y_m)\}$$

$$= \sum_{n=-\infty}^{\infty} E\{X_n^2\}E\{h(t+\tau-n\Delta_t, Y_n)h(t-n\Delta_t, Y_n)\}$$

$$+ \sum_{n,m=-\infty}^{\infty} E\{X_n\}E\{X_m\}E\{h(t+\tau-n\Delta_t, Y_n)\}E\{h(t-m\Delta_t, Y_m)\}$$

$$- \sum_{n=-\infty}^{\infty} (E\{X_n\})^2 E\{h(t+\tau-n\Delta_t, Y_n)\}E\{h(t-n\Delta_t, Y_n)\}$$

$$= \sum_{n=-\infty}^{\infty} \lambda\Delta_t \int_{-\infty}^{\infty} h(t+\tau-n\Delta_t, y)h(t-n\Delta_t, y) f_Y(y) dy$$

$$+ \sum_{n,m=-\infty}^{\infty} \lambda^2\Delta_t^2 \int_{-\infty}^{\infty} h(t+\tau-n\Delta_t) f_Y(y) dy \int_{-\infty}^{\infty} h(t-m\Delta_t, z) f_Y(z) dz$$

$$- \sum_{n=-\infty}^{\infty} \lambda^2\Delta_t^2 \int_{-\infty}^{\infty} h(t+\tau-n\Delta_t, y)h(t-n\Delta_t, y) f_Y(y) dy$$

and, therefore,

$$R_X(\tau) = \lim_{\Delta_t \to 0} R_{\hat{X}}(\tau)$$

$$= \lambda \iint_{-\infty}^{\infty} h(\tau+u, y)h(u, y) f_Y(y) dy du + \left[\lambda \iint_{-\infty}^{\infty} h(u, y) f_Y(y) dy du\right]^2,$$

or

$$K_X(\tau) = R_X(\tau) - m_X^2 = \lambda \iint_{-\infty}^{\infty} h(\tau+u, y)h(u, y) f_Y(y) dy.$$

6.16 Since $Y_i = 1$, then $m_Y = 1$ and $\sigma_Y^2 = 0$. Therefore, from (6.38) and (6.39) (with Z_i replaced by Y_i), we have

$$m_W(t) = \lambda t \quad \text{and} \quad K_W(t_1, t_2) = \lambda \min\{t_1, t_2\}.$$

Since $Z(t) = dW(t)/dt$, then

$$m_Z(t) = \frac{d}{dt} m_W(t) = \lambda$$

$$K_Z(t_1, t_2) = \frac{\partial^2}{\partial t_1 \partial t_2} K_W(t_1, t_2) = \lambda \delta(t_1 - t_2).$$

Therefore, for the Poisson-impulse-sampled process, $U(t) = Z(t)X(t)$, in which $X(t)$ is independent of $Z(t)$, we obtain

$$m_U(t) = m_Z(t) m_X(t) = \lambda m_X$$

$$R_U(\tau) = R_Z(\tau) R_X(\tau) = [K_Z(\tau) + m_Z^2] R_X(\tau) = [\lambda \delta(\tau) + \lambda^2] R_X(\tau).$$

Consequently,

$$\begin{aligned} K_U(\tau) &= R_U(\tau) - m_U^2 = \lambda \delta(\tau) R_X(\tau) + \lambda^2 [R_X(\tau) - m_X^2] \\ &= \lambda(\sigma_X^2 + m_X^2)\delta(\tau) + \lambda^2 K_X(\tau). \end{aligned}$$

This result differs from (6.44) because in (6.42) the $\{Z_i\}$ are independent of the $\{T_i\}$, whereas in (6.59) the $\{X(T_i)\}$ are strongly dependent on $\{T_i\}$.

6.17 *a*) Since $[0, t + \Delta_t) = [0, t) \cup [t, t + \Delta_t)$, then from (6.32) we obtain

$$\begin{aligned} &Prob\{N(t + \Delta_t) - N(t) = 0, N(t) - N(0) = 0\} \\ &\qquad = P_{\Delta_t}(0) P_t(0) = Prob\{N(t + \Delta_t) - N(0) = 0\} \\ &\qquad = Prob\{N(t + \Delta_t) = 0\} = P_{t + \Delta_t}(0). \end{aligned}$$

Substituting (6.33) into this equation yields

$$P_{t+\Delta_t}(0) = P_t(0) P_{\Delta_t}(0) = (1 - \lambda \Delta_t) P_t(0), \quad \text{as} \quad \Delta_t \to 0,$$

or, equivalently,

$$\frac{P_{t+\Delta_t}(0) - P_t(0)}{\Delta_t} = -\lambda P_t(0).$$

Therefore,

$$\lim_{\Delta_t \to 0} \frac{P_{t+\Delta_t}(0) - P_t(0)}{\Delta_t} = \frac{d}{dt} P_t(0) = -\lambda P_t(0).$$

The solution to this differential equation is

$$P_t(0) = c e^{-\lambda t}.$$

But $P_0(0) = 1$; thus $c = 1$ and the probability of zero counts in $[0, t)$ is therefore

$$P_t(0) = e^{-\lambda t}.$$

b) Using (6.33) in

$$P_{t+\Delta_t}(n) = \sum_{k=0}^{n} P_t(n-k)P_{\Delta_t}(k)$$

yields

$$P_{t+\Delta_t}(n) = P_t(n)P_{\Delta_t}(0) + P_t(n-1)P_{\Delta_t}(1) + \sum_{k=2}^{n} P_t(n-k)P_{\Delta_t}(k)$$

$$= (1-\lambda\Delta_t)P_t(n) + \lambda\Delta_t P_t(n-1) + 0$$

as $\Delta_t \to 0$ (since $P_{\Delta_t}(k) \to 0$ for $k > 1$ from (6.33)). Thus,

$$\frac{P_{t+\Delta_t}(n) - P_t(n)}{\Delta_t} = \lambda P_t(n-1) - \lambda P_t(n).$$

Hence,

$$\lim_{\Delta_t \to 0} \frac{P_{t+\Delta_t}(n) - P_t(n)}{\Delta_t} = \frac{d}{dt}P_t(n) = \lambda P_t(n-1) - \lambda P_t(n),$$

which is (6.61). To solve (6.61), we can integrate $dP_t(n)$ or we can multiply both sides of (6.61) by $e^{\lambda t}$ to obtain

$$e^{\lambda t}\frac{d}{dt}P_t(n) + \lambda e^{\lambda t}P_t(n) = \lambda e^{\lambda t}P_t(n-1),$$

which leads to

$$\frac{d}{dt}e^{\lambda t}P_t(n) = \lambda e^{\lambda t}P_t(n-1).$$

For $n = 1$, substituting (6.60) into this equation yields

$$\frac{d}{dt}e^{\lambda t}P_t(1) = \lambda$$

or, equivalently,

$$P_t(1) = (\lambda t + c)e^{-\lambda t}.$$

But $P_0(1) = 0$. Thus, the probability of one count in $[0, t)$ is

$$P_t(1) = \lambda t e^{-\lambda t}.$$

Similarly, for $n = 2$, we have

$$\frac{d}{dt}e^{\lambda t}P_t(2) = \lambda e^{\lambda t}P_t(1) = \lambda^2 t$$

or, equivalently,

$$P_t(2) = \left(\frac{\lambda^2 t^2}{2} + c\right)e^{-\lambda t} = \frac{(\lambda t)^2}{2}e^{-\lambda t}.$$

Therefore, by induction, we obtain

$$\frac{d}{dt}e^{\lambda t}P_t(n) = \frac{\lambda^n t^{n-1}}{(n-1)!}$$

or, equivalently,

$$P_t(n) = \left(\frac{\lambda^n t^n}{n!} + c\right)e^{-\lambda t} = \frac{(\lambda t)^n}{n!}e^{-\lambda t}$$

for the probability of n counts in $[0, t)$.

6.18 *a*) Since, from (6.31), we have

$$Prob\,\{N(t)=n\} = P_t(n)$$

and, therefore,

$$Prob\,\{N(t)<k\} = \sum_{i=0}^{k-1} Prob\,\{N(t)=i\} = \sum_{i=0}^{k-1} P_t(i),$$

then

$$F_{T_k}(t) = 1 - Prob\,\{N(t)<k\} = 1 - \sum_{i=0}^{k-1} P_t(i), \qquad t \geq 0.$$

To obtain the probability density function for the event point T_k, we differentiate $F_{T_k}(t)$ with respect to t:

$$\begin{aligned} f_{T_k}(t) = \frac{dF_{T_k}(t)}{dt} &= \frac{d}{dt}\left[1 - e^{-\lambda t} - \sum_{i=1}^{k-1} e^{-\lambda t}\frac{(\lambda t)^i}{i!}\right] \\ &= \lambda e^{-\lambda t} + \sum_{i=1}^{k-1}\left[\lambda e^{-\lambda t}\frac{(\lambda t)^i}{i!} - \lambda e^{-\lambda t}\frac{(\lambda t)^{i-1}}{(i-1)!}\right] \\ &= \lambda e^{-\lambda t}\frac{(\lambda t)^{k-1}}{(k-1)!}, \qquad t \geq 0, \end{aligned}$$

since all terms for $i > 1$ cancel each other. (The probability densities in this family are called the Gamma densities.)

b) From part *a*, we have

$$E\{T_k\} = \int_{-\infty}^{\infty} t f_{T_k}(t)dt = \int_{0}^{\infty} t\lambda e^{-\lambda t}\frac{(\lambda t)^{k-1}}{(k-1)!}dt.$$

Using integration by parts, we obtain

$$E\{T_k\} = -\frac{\lambda^{k-1}t^k}{(k-1)!}e^{-\lambda t}\Big|_0^\infty + k\int_0^\infty e^{-\lambda t}\frac{(\lambda t)^{k-1}}{(k-1)!}dt = -0+0+\frac{k}{\lambda}\int_0^\infty f_{T_k}(t)dt = \frac{k}{\lambda}.$$

Similarly,

$$E\{T_k^2\} = \int_0^\infty t^2 f_{T_k}(t)dt = \int_0^\infty t^2\lambda e^{-\lambda t}\frac{(\lambda t)^{k-1}}{(k-1)!}dt$$

$$= -\frac{\lambda^{k-1}t^{k+1}}{(k-1)!}e^{-\lambda t}\Big|_0^\infty + (k+1)\int_0^\infty te^{-\lambda t}\frac{(\lambda t)^{k-1}}{(k-1)!}dt$$

$$= -0+0+\frac{k+1}{\lambda}\frac{k}{\lambda} = \frac{k(k+1)}{\lambda^2}.$$

Hence,

$$Var\{T_k\} = E\{T_k^2\} - (E\{T_k\})^2 = \frac{k(k+1)}{\lambda^2} - \left[\frac{k}{\lambda}\right]^2 = \frac{k}{\lambda^2}.$$

6.19 For the interarrival time S_k, since

$$P\{S_k > s\} = P\{T_k - T_{k-1} > s\} = P\{\text{no counts in the interval } (0, s)\} = P_s(0),$$

then

$$F_{S_k}(s) = 1 - P\{S_k > s\} = 1 - P_s(0).$$

The density function is therefore given by

$$f_{S_k}(s) = \frac{dF_{S_k}(s)}{ds} = -\frac{d}{ds}e^{-\lambda s} = \lambda e^{-\lambda s}, \qquad s \ge 0.$$

6.20 Since $\{S_j: j=1, 2, \ldots\}$ is a sequence of independent identically distributed random variables, using the result (1.46) repeatedly yields

$$f_{T_k}(t) = f_S(t) \otimes f_S(t) \otimes \cdots \otimes f_S(t), \qquad t \ge 0.$$

Applying the convolution theorem for the Fourier transform to this equation yields

$$\Phi_{T_k}(\omega) = \int_{-\infty}^{\infty} f_{T_k}(t)e^{i\omega t}dt = \prod_{i=1}^{k}\Phi_S(\omega) = [\frac{\lambda}{\lambda + i\omega}]^k.$$

The inverse Fourier transform of this characteristic function is given by

$$f_{T_k}(t) = \lambda e^{-\lambda t}\frac{(\lambda t)^{k-1}}{(k-1)!}, \qquad t \ge 0,$$

which is the same as (6.63). This inverse Fourier transform can be obtained by using the facts that

$$[\frac{\lambda}{\lambda + i\omega}]^k = \frac{\lambda^k}{(k-1)!(i)^{k-1}} \frac{d^{k-1}}{d\omega^{k-1}}[\frac{1}{\lambda + i\omega}]$$

and

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{d^n}{d\omega^n} F^*(\omega) e^{-i\omega t}\, d\omega = (i)^n t^n f(t)$$

and

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1}{\lambda - i\omega} e^{-i\omega t}\, d\omega = e^{-\lambda t}, \qquad t \geq 0.$$

6.21 The mean and variance of the interarrival time

$$S_k = T_k - T_{k-1}$$

are given by (using (6.65))

$$E\{S_k\} = \int_{-\infty}^{\infty} s f_{S_k}(s) ds = \int_{0}^{\infty} s\lambda e^{-\lambda s} ds = \frac{1}{\lambda}$$

and

$$Var\{S_k\} = E\{S_k^2\} - (E\{S_k\})^2 = \int_0^{\infty} s^2 \lambda e^{-\lambda s} ds - \frac{1}{\lambda^2} = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}.$$

The mean and variance of the time of the kth count are given by (6.64):

$$E\{T_k\} = \frac{k}{\lambda} \quad \text{and} \quad Var\{T_k\} = \frac{k}{\lambda^2}.$$

Therefore, when $\lambda = \lambda_0$, we have

$$E\{S_k\} = \frac{1}{\lambda_0}, \qquad Var\{S_k\} = \frac{1}{\lambda_0^2}$$

$$E\{T_k\} = \frac{k}{\lambda_0}, \qquad Var\{T_k\} = \frac{k}{\lambda_0^2};$$

and when $\lambda = 2\lambda_0$, we have

$$E\{S_k\} = \frac{1}{2\lambda_0}, \qquad Var\{S_k\} = \frac{1}{4\lambda_0^2}$$

$$E\{T_k\} = \frac{k}{2\lambda_0}, \qquad Var\{T_k\} = \frac{k}{4\lambda_0^2}.$$

Hence, the means are both reduced to half their previous values and the variances are reduced to one fourth of their previous values when the rate parameter λ_0 is doubled.

6.22 Since $\lambda(t)$ satisfies

$$\lambda(t)\Delta_t = P\{T < t + \Delta_t \,|T \ge t\} = \frac{P\{t \le T < t + \Delta_t\}}{P\{T \ge t\}}$$
$$= \frac{F_T(t+\Delta_t) - F_T(t)}{1 - F_T(t)},$$

then

$$\lambda(t) = \lim_{\Delta_t \to 0} \frac{F_T(t+\Delta_t) - F_T(t)}{\Delta_t[1 - F_T(t)]} = \frac{dF_T(t)/dt}{1 - F_T(t)}.$$

The solution for $F_T(t)$ can be obtained from (6.66) by integrating

$$\frac{dF_T(t)}{1 - F_T(t)} = \lambda(t)dt,$$

which yields

$$\ln[1 - F_T(t)] = -\int_0^t \lambda(\tau)d\tau + c,$$

where c is an arbitrary constant. For $F_T(0) = 0$, we have $c = 0$ and the above becomes

$$1 - F_T(t) = \exp\{-\int_0^t \lambda(\tau)d\tau\},$$

which results in

$$F_T(t) = 1 - \exp\{-\int_0^t \lambda(\tau)d\tau\}.$$

When $\lambda(t)$ is a constant, we obtain

$$F_T(t) = 1 - e^{-\lambda t}, \qquad t \ge 0$$

and

$$f_T(t) = \frac{dF_T(t)}{dt} = \lambda e^{-\lambda t}, \qquad t \ge 0,$$

which is the same as $f_{T_k}(t)$ with $k = 1$ in (6.63).

6.23 *a*) From (6.66), we have

$$\frac{dF_T(t)/dt}{1 - F_T(t)} = \lambda(t) = kt, \qquad t \ge 0.$$

The solution to this equation with $F_T(0) = 0$ can be obtained by integrating $dF_T(t)$ and is given by

$$F_T(t) = 1 - e^{-kt^2/2}, \qquad t \geq 0.$$

Therefore,

$$f_T(t) = \frac{dF_T(t)}{dt} = kte^{-kt^2/2}, \qquad t \geq 0.$$

b) Again from (6.66) with $\lambda(t) = \gamma t^{\beta}$, we have

$$\frac{dF_T(t)/dt}{1 - F_T(t)} = \gamma t^{\beta}, \qquad t \geq 0,$$

which has the solution

$$F_T(t) = 1 - \exp\{-\gamma t^{\beta+1}/(\beta+1)\}, \qquad t \geq 0.$$

The corresponding density function is, therefore,

$$f_T(t) = \frac{dF_T(t)}{dt} = \gamma t^{\beta}\exp\{-\gamma t^{\beta+1}/(\beta+1)\}, \qquad t \geq 0.$$

6.24 *a*) For this single-server queue, we have

$$\begin{aligned} E\{N(t+S_i)-N(t)\} &= E\{E\{N(t+S_i)-N(t)|S_i\}\} \\ &= E\{\lambda S_i\} = \lambda E\{S_i\} = \lambda m_S. \end{aligned}$$

b) We also have

$$\begin{aligned} m_N &= E\{N_i\} = E\{N(D_i)-N(A_i)\} = E\{E\{N(D_i)-N(A_i)|A_i, D_i\}\} \\ &= E\{\lambda(D_i - A_i)\} = E\{\lambda Q_i\} = \lambda m_Q \end{aligned}$$

by using (6.71). But, from (6.72), we have $m_Q = m_W + m_S$. Therefore,

$$m_N = \lambda(m_W + m_S).$$

c) It follows from (6.74) and (6.75) that

$$m_N = \lambda m_S + \frac{(\lambda m_S)^2 + \lambda^2\sigma_S^2}{2 - 2\lambda m_S} \simeq \begin{cases} \dfrac{1}{2}\lambda^2\sigma_S^2, & \text{small } \lambda m_S \\ \dfrac{1}{2}\lambda m_S, & \text{large } \lambda m_S. \end{cases}$$

Chapter 7

Stochastic Calculus

7.1 For the sampled-and-held process, we have (from exercise 4.10) the autocorrelation

$$R_X(t_1, t_2) = K_X(t_1, t_2) = \sum_{n=-\infty}^{\infty} \sigma_V^2 h(t_1 - nT) h(t_2 - nT),$$

where

$$h(t) = \begin{cases} 1, & 0 \le t < T \\ 0, & otherwise. \end{cases}$$

We can see that condition (7.9) is not satisfied because, for example,

$$\lim_{\varepsilon_1, \varepsilon_2 \to 0} [R_X(t - \varepsilon_1, t - \varepsilon_2) - R_X(t, t)] = 0 - \sigma_V^2 \neq 0$$

at $t = iT$ for i an integer and $\varepsilon_1 = -\varepsilon_2 = \varepsilon$; therefore, $X(t)$ is not mean-square continuous. Similarly, we can see that condition (7.17) is not satisfied because, for example,

$$\frac{\partial^2}{\partial t_1 \partial t_2} R_X(t_1, t_2)\Big|_{t_1 = t_2 = t} = \lim_{\varepsilon_1, \varepsilon_2 \to 0} \frac{1}{\varepsilon_1 \varepsilon_2} \{R_X(t - \varepsilon_1, t - \varepsilon_2) - R_X(t, t - \varepsilon_2)$$

$$- R_X(t - \varepsilon_1, t) + R_X(t, t)\}$$

$$= \frac{1}{0^2} \{\sigma_V^2 - 0 - 0 + \sigma_V^2\} = \infty$$

at $t = iT$ and $\varepsilon_1 = \varepsilon_2 = \varepsilon$; therefore, $X(t)$ is not mean-square differentiable in the narrow sense. However, in the broad sense the derivative can be expressed as (using $h(t) = u(t) - u(t - T)$)

$$\frac{\partial^2}{\partial t_1 \partial t_2} R_X(t_1, t_2) = \sum_{n=-\infty}^{\infty} \sigma_V^2 [\delta(t_1 - nT) - \delta(t_1 - T - nT)][\delta(t_2 - nT) - \delta(t_2 - T - nT)],$$

where $\delta(\cdot)$ is the Dirac delta.

7.2 For the phase-randomized sampled-and-held process, we have (from Section 4.2.7) the autocorrelation

$$R_X(t_1, t_2) = K_X(t_1, t_2) = \frac{\sigma_V^2}{T} r_h(t_1 - t_2).$$

We can see that $X(t)$ is mean-square continuous because condition (7.9) is satisfied:

$$\lim_{\varepsilon_1, \varepsilon_2 \to 0} [R_X(t-\varepsilon_1, t-\varepsilon_2) - R_X(t, t)] = \lim_{\varepsilon_1, \varepsilon_2 \to 0} \frac{\sigma_V^2}{T}[r_h(\varepsilon_2 - \varepsilon_1) - r_h(0)] = 0$$

because $r_h(\tau)$ is continuous at $\tau = 0$. Since $X(t)$ is a WSS process, condition (7.19) is used to examine the mean-square differentiability. From (7.19) we obtain

$$\frac{d^2}{d\tau^2}R_X(\tau) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon^2}[R_X(\varepsilon) - 2R_X(0) + R_X(-\varepsilon)] = \lim_{\varepsilon \to 0} \frac{2\sigma_V^2}{T}\frac{1}{\varepsilon^2}[r_h(\varepsilon) - r_h(0)]$$

$$= \lim_{\varepsilon \to 0} \frac{2\sigma_V^2}{T}\frac{1}{\varepsilon^2}[T - \varepsilon - T] = -\infty.$$

That is, this derivative does not exist. But if the Dirac delta function is used, we obtain

$$\frac{d^2}{d\tau^2}R_X(\tau) = \frac{d}{d\tau}[\frac{d}{d\tau}\frac{2\sigma_V^2}{T}r_h(\tau)] = \frac{\sigma_V^2}{T}[\delta(\tau+T) - 2\delta(\tau) + \delta(\tau - T)].$$

Thus, the derivative exists in the broad sense.

7.3 *a*) Using the definition of $Z(t)$ in Chapter 6 yields

$$E\{W(t)\frac{dW(t)}{dt}\} = E\{\int_{-\infty}^{\infty} h(t-\tau)Z(\tau)d\tau\, Z(t)\} = \int_{-\infty}^{\infty} h(t-\tau)E\{Z(\tau)Z(t)\}d\tau$$

$$= \int_{-\infty}^{\infty} h(t-\tau)\alpha^2\delta(\tau - t)d\tau = \alpha^2 \int_0^t \delta(\tau - t)d\tau, \qquad t \ge 0.$$

b) The left side of the equation in this exercise can be reexpressed as

$$R \triangleq E\{X(t)\frac{dX(t)}{dt}\} = E\{\frac{d}{dt}X(t)X(s)\}\Big|_{s=t} = \left[\frac{d}{dt}R_X(t-s)\right]_{s=t}.$$

Let $t - s = \tau$; then $dt = d\tau$ and the preceding becomes

$$R = \frac{d}{d\tau}R_X(\tau)\Big|_{\tau=0}.$$

But since $R_X(\tau) = R_X(-\tau)$, then

$$\frac{d}{d\tau}R_X(\tau) = \frac{d}{d\tau}R_X(-\tau) = -\frac{d}{d\sigma}R_X(\sigma)\Big|_{\sigma=-\tau}$$

and, therefore,

$$R = \frac{d}{d\tau}R_X(\tau)\Big|_{\tau=0} = -\frac{d}{d\tau}R_X(\tau)\Big|_{\tau=0} = -R,$$

which implies that $R = 0$.

7.4 Since $X(t)$ is a WSS integrable process, then we have

$$E\{X(t)\} = m_X$$

$$E\{(X(t_1)-m_X)(X(t_2)-m_X)\} = K_X(t_1-t_2)$$

and the integrated process

$$Y(t) = \int_0^t X(u)du, \qquad t > 0$$

has mean and variance given by

$$m_Y(t) = E\{Y(t)\} = \int_0^t E\{X(u)\}du = \int_0^t m_X du = m_X t, \qquad t > 0$$

and

$$\sigma_Y^2(t) = E\{[Y(t)-m_Y(t)]^2\} = \int_0^t\int_0^t E\{[X(t_1)-m_X][X(t_2)-m_X]\}dt_1dt_2$$

$$= \int_0^t\int_0^t K_X(t_1-t_2)dt_1dt_2, \qquad t > 0.$$

By making the change of variables $\tau = t_1 - t_2$ and using the window function

$$h(u) = \begin{cases} 1, & 0 \le u < t \\ 0, & otherwise, \end{cases}$$

$\sigma_Y^2(t)$ can be further expressed as

$$\sigma_Y^2(t) = \iint_{-\infty}^{\infty} h(t_1)h(t_2)K_X(t_1-t_2)dt_1dt_2 = \int_{-\infty}^{\infty} K_X(\tau)\int_{-\infty}^{\infty} h(t_2+\tau)h(t_2)dt_2d\tau$$

$$= \int_{-\infty}^{\infty} K_X(\tau)r_h(\tau)d\tau = t\int_{-t}^{t} (1-\frac{|\tau|}{t})K_X(\tau)d\tau, \qquad t > 0,$$

since

$$r_h(\tau) = \begin{cases} \int_{-\infty}^{\infty} h(\sigma+\tau)h(\sigma)d\sigma = t - |\tau|, & |\tau| \le t, \\ 0, & |\tau| > t. \end{cases}$$

7.5 Since

$$0 \le [E\{X(t)-X(t-\varepsilon)\}]^2 \le E\{[X(t)-X(t-\varepsilon)]^2\}$$

and since

$$\lim_{\varepsilon \to 0} E\{[X(t)-X(t-\varepsilon)]^2\} = 0$$

for a mean-square continuous process $X(t)$, then

$$\lim_{\varepsilon \to 0} [E\{X(t)-X(t-\varepsilon)\}]^2 = \lim_{\varepsilon \to 0} [m_X(t)-m_X(t-\varepsilon)]^2 = 0,$$

which implies that

$$\lim_{\varepsilon \to 0} [m_X(t)-m_X(t-\varepsilon)] = 0.$$

Thus, the mean function $m_X(t)$ is continuous at t.

7.6 From condition (7.16), we have

$$\begin{aligned} 0 &= \lim_{\varepsilon \to 0} E\{(\frac{1}{\varepsilon}[X(t)-X(t-\varepsilon)]-X^{(1)}(t))^2\} \\ &= \lim_{\varepsilon \to 0} (\frac{1}{\varepsilon^2}E\{[X(t)-X(t-\varepsilon)]^2\} - \frac{2}{\varepsilon}E\{[X(t)-X(t-\varepsilon)]X^{(1)}(t)\} \qquad (*) \\ &\qquad + E\{[X^{(1)}(t)]^2\}) \end{aligned}$$

for a mean-square differentiable process $X(t)$. Also, from the Cauchy-Schwarz inequality, we have

$$|\frac{1}{\varepsilon}E\{[X(t)-X(t-\varepsilon)]X^{(1)}(t)\}| \le \frac{1}{\varepsilon}(E\{[X(t)-X(t-\varepsilon)]^2\})^{1/2}(E\{[X^{(1)}(t)]^2\})^{1/2}.$$

Thus, (*) can be valid only if the quantity

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon^2}E\{[X(t)-X(t-\varepsilon)]^2\}$$

is finite. This is a sufficient condition for

$$\lim_{\varepsilon \to 0} E\{[X(t)-X(t-\varepsilon)]^2\} = 0$$

(which is condition (7.8)) to be valid. Thus, if $X(t)$ is mean-square differentiable, then it must be mean-square continuous.

7.7 For the phase-randomized sampled-and-held process $Y(t)$, we have (from Section 4.2.7) the covariance

$$K_Y(\tau) = \frac{\sigma_V^2}{T} r_h(\tau),$$

where

$$r_h(\tau) = \begin{cases} T(1-|\tau|/T), & |\tau| \le T \\ 0, & \textit{otherwise}. \end{cases}$$

It follows that

$$\frac{\partial}{\partial \tau} K_Y(\tau) = \begin{cases} \sigma_V^2/T, & -T \le \tau \le 0 \\ -\sigma_V^2/T, & 0 < \tau \le T \\ 0, & otherwise \end{cases}$$

and, therefore, that the covariance of the differentiated process $Y^{(1)}(t)$ is given by (using (7.23))

$$K_{Y^{(1)}}(\tau) = -\frac{\partial^2}{\partial \tau^2} K_Y(\tau) = -2\frac{\sigma_V^2}{T}\delta(\tau).$$

7.8 For the random-amplitude-and-phase sine wave process $Y(t)$, we have (from Section 4.2.5) the autocorrelation

$$R_Y(t_1, t_2) = \frac{1}{2} E\{A^2\} \cos(\omega_0[t_1 - t_2]).$$

It follows from (7.28) that the autocorrelation for the integrated process $Y^{(-1)}(t)$ is given by

$$R_{Y^{(-1)}}(t_1, t_2) = \int_0^{t_1}\int_0^{t_2} R_Y(u_1, u_2)du_1 du_2 = \frac{1}{2\omega_0} E\{A^2\} \int_0^{t_2} [\sin(\omega_0[t_1 - u_2]) + \sin(\omega_0 u_2)] du_2$$

$$= \frac{1}{2\omega_0^2} E\{A^2\}[\cos(\omega_0[t_1 - t_2]) - \cos(\omega_0 t_1) - \cos(\omega_0 t_2) + 1].$$

Consequently, $Y^{(-1)}$ is a nonstationary process. Observe that

$$Y^{(-1)}(t) = \frac{A}{\omega_0}[\sin(\omega_0 t + \theta) - \sin\theta];$$

thus, we see that the sum of two correlated processes ($\frac{A}{\omega_0}\sin(\omega_0 t + \theta)$ and $-\sin\theta$), each of which is individually stationary, can be nonstationary.

7.9 It follows from definition (7.37) that

$$E\{Y_1 Y_2^*\} = E\{\int_0^T \phi_1(t) Z(t) dt \int_0^T \phi_2^*(u) Z(u) du\} = \int_0^T\int_0^T \phi_1(t)\phi_2^*(u) E\{Z(t)Z(u)\} du dt$$

$$= \int_0^T\int_0^T \phi_1(t)\phi_2^*(u)\alpha^2\delta(t-u) du dt = \alpha^2 \int_0^T \phi_1(t)\phi_2^*(t) dt.$$

Thus, if ϕ_1 and ϕ_2 are orthogonal functions, then Y_1 and Y_2 are orthogonal random variables.

7.10 *a*) Taking the expected value of both sides of the equation given in this exercise and interchanging the order of operations yields

$$\frac{d}{dt} m_X(t) = a m_X(t) + m_Z, \qquad t \geq 0.$$

The solution to this differential equation with the initial condition $m_X(0) = 1$ is

$$m_X(t) = \left(1 + \frac{m_Z}{a}\right) e^{at} - \frac{m_Z}{a}, \qquad t \geq 0.$$

b) Taking the expected value of both sides of the equation given in this exercise and interchanging the order of operations yields

$$\int_0^t m_X(u) du = b m_X(t), \qquad t \geq 0.$$

The solution to this integral equation is

$$m_X(t) = 0, \qquad t \geq 0.$$

7.11 The relationship between $X(t)$ and $Y(t)$ is given by

$$RY(t) + \frac{1}{C} \int_0^t Y(u) du + L \frac{dY(t)}{dt} = X(t).$$

Assuming that $Y(t)$ and $X(t)$ are mean-square integrable and differentiable, we differentiate this equation and take the expected value of both sides to obtain the following differential equation for the mean $m_Y(t)$:

$$L \frac{d^2}{dt^2} m_Y(t) + R \frac{d}{dt} m_Y(t) + \frac{1}{C} m_Y(t) = \frac{d}{dt} m_X(t).$$

7.12 The mean and autocorrelation of the given process $V(t)$ are obtained as follows:

$$m_V(t) = E\{V(t)\} = \int_{t-T}^{t} E\{W(s)\} ds = 0$$

and

$$R_V(t_1, t_2) = E\{ \int_{t_1-T}^{t_1} W(s) ds \int_{t_2-T}^{t_2} W(t) dt \} = \int_{t_1-T}^{t_1} \int_{t_2-T}^{t_2} E\{W(s)W(t)\} ds dt$$

$$= \alpha^2 \int_{t_1-T}^{t_1} \int_{t_2-T}^{t_2} \min\{s, t\} ds dt = \alpha^2 \int_{t_1-T}^{t_1} \int_{t_2-T}^{t_2} [s u(t-s) + t u(s-t)] ds dt$$

$$= \frac{1}{6}(t_1^3 - t_2^3) + \frac{T}{2}(t_1^2 + t_2^2) - \frac{1}{2} t_1 t_2 (t_1 - t_2 + 2T) + \frac{T^2}{2}(t_1 + t_2) - \frac{2}{3} T^3, \quad t_1 \geq t_2.$$

For $t_2 > t_1$, we simply interchange t_1 and t_2 in this equation.

7.13 *a*) For the Ornstein-Uhlenbeck process $X(t)$, we have

$$R_X(\tau) = \sigma^2 e^{-\alpha|\tau|}.$$

The autocorrelation function for the smoothed process is given by

$$R_Y(t+\tau, t) = E\{\frac{1}{\Delta}\int_{t+\tau-\Delta}^{t+\tau} X(u)du \ \frac{1}{\Delta}\int_{t-\Delta}^{t} X(v)dv\}$$

$$= \frac{1}{\Delta^2}\int_{t+\tau-\Delta}^{t+\tau}\int_{t-\Delta}^{t} E\{X(u)X(v)\}dudv = \frac{1}{\Delta^2}\int_{t+\tau-\Delta}^{t+\tau}\int_{t-\Delta}^{t} R_X(u-v)dudv.$$

Using the change of variables $u = s - (t+\tau-\Delta)$ and $v = w - (t-\Delta)$ yields

$$R_Y(t+\tau, t) = \frac{1}{\Delta^2}\int_0^{\Delta}\int_0^{\Delta} R_X(s-w-\tau)dsdw = R_Y(\tau).$$

It follows from (7.34) that

$$R_Y(\tau) = \frac{1}{\Delta}\int_{-\Delta}^{\Delta}(1-\frac{|z|}{\Delta})R_X(z-\tau)dz = \frac{\sigma^2}{\Delta}\int_{-\Delta}^{\Delta}(1-\frac{|z|}{\Delta})e^{-\alpha|z-\tau|}dz.$$

For $0 \le \tau < \Delta$,

$$R_Y(\tau) = \frac{\sigma^2}{\Delta}[\int_{-\Delta}^{0}(1+\frac{z}{\Delta})e^{\alpha(z-\tau)}dz + \int_0^{\tau}(1-\frac{z}{\Delta})e^{\alpha(z-\tau)}dz + \int_{\tau}^{\Delta}(1-\frac{z}{\Delta})e^{-\alpha(z-\tau)}dz].$$

Integration by parts yields

$$R_Y(\tau) = (\frac{\sigma}{\Delta\alpha})^2[2\alpha(\Delta-\tau) + e^{\alpha(\tau-\Delta)} - e^{-\alpha(\tau+\Delta)} - 2e^{-\alpha\tau}].$$

For $\tau \ge \Delta$,

$$R_Y(\tau) = \frac{\sigma^2}{\Delta}[\int_{-\Delta}^{0}(1+\frac{z}{\Delta})e^{\alpha(z-\tau)}dz + \int_0^{\Delta}(1-\frac{z}{\Delta})e^{\alpha(z-\tau)}dz]$$

$$= (\frac{\sigma}{\Delta\alpha})^2[e^{-\alpha(\tau-\Delta)} + e^{-\alpha(\tau+\Delta)} - 2e^{-\alpha\tau}].$$

b) The mean-square value of $Y(t)$ will approach the squared mean value of $X(t)$ (which is zero) as Δ is increased. To see this, we use the result of part *a* for $R_Y(0)$ to obtain

$$E\{Y^2(t)\} = 2(\frac{\sigma}{\Delta\alpha})^2(\alpha\Delta + e^{-\alpha\Delta} - 1) \to \frac{2\sigma^2}{\alpha\Delta} \to 0 \quad \text{as} \quad \Delta \to \infty.$$

7.14 From the given autocorrelation of $X(t)$, we know that $E\{X^2(t)\} = 10^{-8}$. Therefore, we can use the approximation $\sin[X(t)] \simeq X(t)$ to obtain

$$Y(t) \simeq g \int_0^t X(u)du.$$

Thus, the mean-square error is given by

$$MSE \triangleq E\{Y^2(t)\} = g^2 \int_0^t \int_0^t E\{X(u)X(v)\}dudv = g^2 \int_0^t \int_0^t R_X(u-v)dudv$$

$$= g^2 t \int_{-t}^{t} (1 - \frac{|\tau|}{t}) R_X(\tau) d\tau = 10^{-8} g^2 t \int_{-t}^{t} (1 - \frac{|\tau|}{t}) e^{-|\tau|/100} d\tau$$

$$= 2 \times 10^{-8} g^2 t \int_0^t (1 - \frac{\tau}{t}) e^{-\tau/100} d\tau.$$

Using integration by parts yields

$$MSE = 2 \times 10^{-8} g^2 t \left[-100 e^{-\tau/100} \Big|_0^t + \frac{100}{t} \tau e^{-\tau/100} \Big|_0^t + \frac{(100)^2}{t} e^{-\tau/100} \Big|_0^t \right]$$

$$= 2 \times 10^{-6} g^2 t \left[1 + \frac{100}{t} (e^{-t/100} - 1) \right].$$

For $t = 12$ hours $= 43{,}200$ seconds, the mean-square error is

$$MSE \simeq 8.3 \quad \text{m}^2/\text{s}^2,$$

and it grows linearly with t for $t \gg 100$.

7.15 *a*) The autocorrelation of $X^{(-1)}(t)$ is given by

$$R_{X^{(-1)}}(\tau) = E\{X^{(-1)}(t+\tau) X^{(-1)}(t)\} = \int_{-\infty}^{t+\tau} \int_{-\infty}^{t} E\{X(u)X(v)\}dvdu$$

$$= \int_{-\infty}^{t+\tau} \int_{-\infty}^{t} R_X(u-v)dvdu.$$

Making the changes of variables $\sigma = u - t$ and then $\mu = \sigma + t - v$ yields

$$R_{X^{(-1)}}(\tau) = \int_{-\infty}^{\tau} \int_{-\infty}^{t} R_X(\sigma + t - v)dvd\sigma = -\int_{-\infty}^{\tau} \int_{\infty}^{\sigma} R_X(\mu)d\mu d\sigma = \int_{-\infty}^{\tau} \int_{\sigma}^{\infty} R_X(\mu)d\mu d\sigma,$$

which is the desired result (7.40).

b) For $R_X(\tau) = \delta(\tau)$, we have

$$R_{X^{(-1)}}(\tau) = \int_{-\infty}^{\tau} u(-\sigma)d\sigma = \infty.$$

For $R_X(\tau) = e^{-|\tau|}$, we have

$$R_{X^{(-1)}}(\tau) = \int_{-\infty}^{\tau} [2u(-\sigma) + e^{-\sigma}u(\sigma) - e^{\sigma}u(-\sigma)]d\sigma = \infty.$$

For $R_X(\tau) = 2\delta(\tau) - e^{-|\tau|}$, we have

$$R_{X^{(-1)}}(\tau) = \int_{-\infty}^{\tau} [e^{-\sigma}u(\sigma) - e^{\sigma}u(-\sigma)]d\sigma < \infty.$$

7.16 *a*) Since

$$\delta[X(t) - x] = \left|\frac{dX(t)}{dt}\right|^{-1} \sum_i \delta(t - t_i),$$

then the number of zero crossings of $X(t)$ for $u \le t \le v$ is

$$N_X(u, v) = \int_u^v \sum_i \delta(t - t_i)dt = \int_u^v \delta[X(t) - x]Y(t)dt.$$

Taking the expected value of $N_X(u, v)$ yields

$$E\{N_X(u, v)\} = \int_u^v E\{\delta[X(t) - x]Y(t)\}dt = \int_u^v [\iint_{-\infty}^{\infty} \delta(u - x)vf_{Y(t)X(t)}(v, u)dudv]\, dt$$

$$= \int_u^v [\iint_{-\infty}^{\infty} \delta(u - x)vf_{Y(t)|X(t)}(v\,|u)f_{X(t)}(u)dudv]\, dt$$

$$= \int_u^v \int_{-\infty}^{\infty} vf_{Y(t)|X(t)}(v\,|x)f_{X(t)}(x)dv\ \ dt$$

$$= \int_u^v f_{X(t)}(x)E\{Y(t)|X(t) = x\}dt,$$

which is the desired result (7.41).

b) If $X(t)$ is stationary, then $f_{X(t)}(x)$ and $E\{Y(t)|X(t) = x\}$ are independent of t. Therefore,

$$E\{N_X(u, v)\} = (v - u)f_{X(t)}(x)E\{Y(t)|X(t) = x\}.$$

c) Since X and dX/dt are jointly Gaussian, then their uncorrelatedness guarantees their independence. Therefore,

$$E\{N_X(u, v)\} = (v - u)f_{X(t)}(x)E\{|\frac{dX(t)}{dt}|\},$$

where,

$$E\{|\frac{dX(t)}{dt}|\} = \int_{-\infty}^{\infty} |z| f_Z(z)dz = 2\int_{0}^{\infty} z f_Z(z)dz$$

$$= 2\int_{0}^{\infty} \frac{z}{\sigma\sqrt{2\pi}} \exp\{-\frac{z^2}{2\sigma^2}\}dz = \sigma\left[\frac{2}{\pi}\right]^{1/2},$$

for which $Z = dX(t)/dt$ and

$$\sigma = \left[R_Z(0)\right]^{1/2} = \left[-\frac{d^2}{d\tau^2}R_X(\tau)\Big|_{\tau=0}\right]^{1/2}.$$

The desired result (7.42) follows.

7.17 For $Y(t) = X(t) + V(t)$ and $\int_{-\infty}^{\infty} V^2(t)dt < \infty$, we have

$$\lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} [X(t)-Y(t)]^2 dt = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} V^2(t)dt = \lim_{T\to\infty} \frac{1}{T} \int_{-\infty}^{\infty} V^2(t)dt = 0.$$

Thus, (7.43) is valid and $X(t)$ and $Y(t)$ are, therefore, temporally mean-square equivalent.

Chapter 8

Ergodicity and Duality

8.1 It follows from (8.17) and (8.18) that the mean of the estimate of the mean is equal to the mean being estimated,

$$E\{\hat{m}_X^N\} = E\{\frac{1}{N}\sum_{i=1}^{N} X(i\Delta)\} = \frac{1}{N}\sum_{i=1}^{N} E\{X(i\Delta)\} = \frac{1}{N}\sum_{i=1}^{N} m_X = m_X,$$

and the variance of the estimate of the mean is given by

$$Var\{\hat{m}_X^N\} = E\{[\hat{m}_X^N - E\{\hat{m}_X^N\}]^2\} = E\{\frac{1}{N}\sum_{i=1}^{N}[X(i\Delta)-m_X]\,\frac{1}{N}\sum_{j=1}^{N}[X(j\Delta)-m_X]\}$$

$$= \frac{1}{N^2}\sum_{i=1}^{N}\sum_{j=1}^{N} E\{[X(i\Delta)-m_X][X(j\Delta)-m_X]\} = \frac{1}{N^2}\sum_{i=1}^{N}\sum_{j=1}^{N} K_X([i-j]\Delta).$$

Substituting (8.20) into (8.19) yields the following approximation for the mean squared error:

$$MSE_N \triangleq E\{[m_X - \hat{m}_X^N]^2\} = Var\{\hat{m}_X^N\} = \frac{1}{N^2}\sum_{i=1}^{N}\sum_{j=1}^{N} K_X([i-j]\Delta)$$

$$= \frac{1}{N^2}\left(\sum_{i=1}^{N} K_X(0) + \sum_{\substack{i=1\\ j\neq i}}^{N}\sum_{j=1}^{N} K_X([i-j]\Delta)\right) \simeq \frac{\sigma_X^2}{N}.$$

8.2 Using the window function $h(k)$ defined by

$$h(k) = \begin{cases} 1, & 1 \le k \le N \\ 0, & otherwise \end{cases}$$

and the change of variables $i - j = k$ in (8.19) results in

$$Var\{\hat{m}_X^N\} = \frac{1}{N^2}\sum_{i,j=-\infty}^{\infty} h(i)h(j)K_X([i-j]\Delta) = \frac{1}{N^2}\sum_{j,k=-\infty}^{\infty} h(k+j)h(j)K_X(k\Delta)$$

$$= \frac{1}{N^2}\sum_{k=-\infty}^{\infty} K_X(k\Delta)\sum_{j=-\infty}^{\infty} h(k+j)h(j). \qquad (*)$$

But, since

$$\sum_{j=-\infty}^{\infty} h(k+j)h(j) = \begin{cases} N - |k|, & -N \le k \le N \\ 0, & otherwise, \end{cases}$$

then (*) simplies to (using $N = T/\Delta$)

$$Var\{\hat{m}_X^{T/\Delta}\} = Var\{\hat{m}_X^N\} = \frac{1}{N^2}\sum_{k=-N}^{N}(N - |k|)K_X(k\Delta)$$

$$= \frac{1}{N}K_X(0) + \frac{2}{N}\sum_{k=1}^{N}(1-\frac{k}{N})K_X(k\Delta) = \frac{\sigma_X^2}{N} + \frac{2}{T}\sum_{k=1}^{N}(1-\frac{k\Delta}{T})K_X(k\Delta)\Delta.$$

8.3 Since $K_X(\tau) \ll K_X(0)$ for $\tau > \tau_0$, then

$$(1-\frac{\tau}{T})K_X(\tau) \simeq K_X(\tau), \quad T \gg \tau_0.$$

Thus, the mean squared error is approximated by

$$MSE(T) \triangleq \frac{2}{T}\int_0^T (1-\frac{\tau}{T})K_X(\tau)d\tau \simeq \frac{2}{T}\int_0^T K_X(\tau)d\tau \simeq \frac{2}{T}\sigma_X^2\frac{\tau_0}{2} = \frac{\sigma_X^2}{T/\tau_0},$$

and this verifies (8.25).

8.4 *a*) We can use the Cauchy-Schwarz inequality as follows:

$$|\frac{1}{T}\int_0^T K_X(\tau)d\tau|^2 = |E\{[X(0)-m_X]\frac{1}{T}\int_0^T [X(\tau)-m_X]d\tau\}|^2$$

$$\le E\{[X(0)-m_X]^2\}E\{[\frac{1}{T}\int_0^T [X(\tau)-m_X]d\tau]^2\}$$

$$= \frac{\sigma_X^2}{T^2}\int_0^T\int_0^T K_X(s-\tau)dsd\tau = \frac{\sigma_X^2}{T}\int_{-T}^{T}(1-\frac{|\tau|}{T})K_X(\tau)d\tau$$

$$= 2\frac{\sigma_X^2}{T}\int_0^T (1-\frac{\tau}{T})K_X(\tau)d\tau.$$

Therefore, (8.42c) is necessary for (8.42b).

b) It follows from the first equality in (8.106) that we are integrating a function over the square region $[0, T] \times [0, T]$, and this function is symmetrical about the diagonal of this square. Thus, we will get the same result if we integrate over only the upper (or lower) triangular region above (or below) the diagonal and then multiply the result by 2. This yields the second equality in (8.106). Now, by using the result (8.106) in (8.107)-(8.108), we see that (8.42c) is sufficient for (8.42b).

8.5 For $\Delta = 2\pi/\omega_0$, (8.36) yields

$$K_Y(k\Delta) = \frac{1}{2}E\{A^2\}.$$

Therefore, (8.109) reduces to

$$\frac{1}{2}E\{A^2\} \lim_{N\to\infty} \frac{1}{N}\sum_{k=1}^{N}(1-\frac{k}{N}) = \frac{1}{4}E\{A^2\} \neq 0.$$

The problem here is that (8.18) yields

$$\hat{m}_X^N = \frac{1}{N}\sum_{i=1}^{N} A\sin(i\,2\pi + \Theta) = A\sin(\Theta),$$

which does not converge to $m_X = 0$ as $N \to \infty$. In contrast to this, it is shown in exercise 8.6 that (8.42b) is satisfied and this agrees with the fact that (8.32) has the limit

$$\hat{m}_X(T) = \frac{A}{\omega_0 T}[\cos(\omega_0 T + \Theta) - \cos(\Theta)] \quad \to \quad 0 \quad \text{as} \quad T \to \infty.$$

8.6 Substituting (8.36) into (8.34) yields

$$MSE(T) = \frac{2}{T}\int_0^T (1-\frac{\tau}{T})K_Y(\tau)d\tau = \frac{E\{A^2\}}{T}\int_0^T [\cos(\omega_0\tau) - \frac{\tau}{T}\cos(\omega_0\tau)]d\tau$$

$$= \frac{E\{A^2\}}{T}[\frac{1}{\omega_0}\sin(\omega_0\tau)\Big|_0^T - \frac{1}{T\omega_0}\tau\sin(\omega_0\tau)\Big|_0^T - \frac{1}{T\omega_0^2}\cos(\omega_0\tau)\Big|_0^T]$$

$$= E\{A^2\}[\frac{\sin(\omega_0 T)}{\omega_0 T} - \frac{\sin(\omega_0 T)}{\omega_0 T} - \frac{\cos(\omega_0 T)}{\omega_0^2 T^2} + \frac{1}{\omega_0^2 T^2}].$$

Therefore,

$$\lim_{T\to\infty} MSE(T) = 0$$

and (8.31) is satisfied. However, substituting (8.38) into (8.34) yields

$$MSE(T) = \frac{2}{T}\int_0^T (1-\frac{\tau}{T})\sigma_Z^2 d\tau + \frac{E\{A^2\}}{T}\int_0^T (1-\frac{\tau}{T})\cos(\omega_0\tau)d\tau.$$

Since the second term approaches zero as $T \to \infty$, we obtain

$$\lim_{T\to\infty} MSE(T) = \lim_{T\to\infty}\frac{2}{T}\int_0^T (1-\frac{\tau}{T})\sigma_Z^2 d\tau = \lim_{T\to\infty}\frac{2\sigma_Z^2}{T}[T - \frac{T}{2}] = \sigma_Z^2,$$

which violates (8.31).

8.7 Since $K_X(\tau) = \frac{1}{2}\sigma_A^2\cos(\omega_0\tau)$, it is shown in the first part of exercise 8.6 that

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T (1-\frac{\tau}{T})K_X(\tau)d\tau = 0.$$

Thus, $X(t)$ has mean-ergodicity of the mean. Since $X(t)$ is a Gaussian process, then from (8.52) we have

$$\begin{aligned} K_{Y_\tau}(u) &= K_X^2(u) + K_X(u+\tau)K_X(u-\tau) \\ &= \frac{\sigma_A^4}{4}[\cos^2(\omega_0\tau) + \cos(\omega_0[u+\tau])\cos(\omega_0[u-\tau])] \\ &= \frac{\sigma_A^4}{4}[\frac{1}{2} + \frac{1}{2}\cos(2\omega_0\tau) + \frac{1}{2}\cos(2\omega_0 u) + \frac{1}{2}\cos(2\omega_0\tau)] \\ &= \frac{\sigma_A^4}{8}[1 + 2\cos(2\omega_0\tau) + \cos(2\omega_0 u)]. \end{aligned}$$

Consequently, we obtain

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T K_{Y_\tau}(u)du = \frac{\sigma_A^4}{8}[1 + 2\cos(2\omega_0\tau)] \neq 0.$$

Thus, (8.48b) is violated and $X(t)$, therefore, does not have mean-square ergodicity of the autocorrelation unless $\sigma_A^2 = 0$.

8.8 The empirical and probabilistic mean and autocorrelation of $Y(t)$ are found to be

$$\hat{m}_Y = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} Y(t)dt = A \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} X(t)dt = A\hat{m}_X,$$

$$m_Y = E\{Y(t)\} = E\{AX(t)\} = E\{A\}E\{X(t)\} = E\{A\}m_X,$$

$$\hat{R}_Y(\tau) = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} Y(t+\tau)Y(t)dt = A^2 \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} X(t+\tau)X(t)dt = A^2\hat{R}_X(\tau),$$

$$R_Y(\tau) = E\{Y(t+\tau)Y(t)\} = E\{A^2\}E\{X(t+\tau)X(t)\} = E\{A^2\}R_X(\tau).$$

Although $\hat{m}_X = m_X$ and $\hat{R}_X = R_X$, we see from the above equations that $\hat{m}_Y \neq m_Y$ and $\hat{R}_Y \neq R_Y$ (unless A is nonrandom). Therefore, $Y(t)$ does not exhibit ergodicity of the mean or of the autocorrelation. The autocovariance of $Y(t)$ is given by

$$K_Y(\tau) = R_Y(\tau) - m_Y^2 = E\{A^2\}R_X(\tau) - (E\{A\})^2 m_X^2 = E\{A^2\}K_X(\tau) + \sigma_A^2 m_X^2.$$

Since $X(t)$ exhibits mean-square ergodicity of the mean, then from (8.42c) we have

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T K_Y(\tau)d\tau = E\{A^2\} \lim_{T\to\infty} \frac{1}{T}\int_0^T K_X(\tau)d\tau + \sigma_A^2 m_X^2 = \sigma_A^2 m_X^2 \neq 0.$$

Thus, $Y(t)$ does not exhibit mean-square ergodicity of the mean. The fourth-order moment of $Y(t)$ is given by

$$E\{Y(t+\tau+u)Y(t+u)Y(t+\tau)Y(t)\} - R_Y^2(\tau)$$

$$= E\{A^4\}E\{X(t+\tau+u)X(t+u)X(t+\tau)X(t)\} - (E\{A^2\})^2R_X^2(\tau).$$
$$= E\{A^4\}[E\{X(t+\tau+u)X(t+u)X(t+\tau)X(t)\} - R_X^2(\tau)] + \sigma_{A^2}^2 R_X^2(\tau).$$

Since $X(t)$ exhibits mean-square ergodicity of the autocorrelation, then from (8.48b) we have (using $Y_\tau(t) = Y(t+\tau)Y(t)$)

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T K_{Y_\tau}(u)du$$
$$= E\{A^4\}\lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} [E\{X(t+\tau+u)X(t+u)X(t+\tau)X(t)\} - R_X^2(\tau)]du + \sigma_{A^2}^2 R_X^2(\tau)$$
$$= \sigma_{A^2}^2 R_X^2(\tau) \neq 0.$$

Thus, $Y(t)$ does not exhibit mean-square ergodicity of the autocorrelation.

8.9 *a*) Since $X(t) = 0$ for $t < 0$, it follows from (8.110) that

$$\underset{T\to\infty}{\text{l.i.m.}} \frac{1}{T}\int_{-T/2}^{T/2} X(t)dt = \frac{1}{2}\underset{T/2\to\infty}{\text{l.i.m.}} \frac{1}{T/2}\int_0^{T/2} X(t)dt = \frac{1}{2}E\{X(t)\}.$$

b) Since the Ornstein-Uhlenbeck process is a zero-mean Gaussian process with autocovariance given by

$$K_X(\tau) = \sigma^2 e^{-\alpha|\tau|},$$

then (8.58) can be used to determine the mean-square ergodicity of the autocorrelation. Substituting $K_X(\tau)$ into (8.58) yields

$$\hat{R}_K(0) = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} K_X^2(u)du = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} \sigma^2 e^{-2\alpha|u|}du = \lim_{T\to\infty} \frac{\sigma^2}{T}\frac{1}{2\alpha}[1-e^{-2\alpha T}] = 0.$$

Therefore, the Ornstein-Uhlenbeck process has mean-square ergodicity of the autocorrelation.

8.10 For the given process, we can show that

$$\hat{R}_X(\tau)_T = \hat{R}_A(\tau)_T + \hat{R}_{AB}(\tau)_T + \hat{R}_{BA}(\tau)_T + \hat{R}_B(\tau)_T,$$

where

$$A(t) \triangleq \sum_n A_n \cos(\omega_n t + \Theta_n).$$

Since $B(t)$ has mean square ergodicity of the autocorrelation, then $\hat{R}_B(\tau)_T$ converges to $R_B(\tau)$. Also, we know that $B(t)$ cannot contain any finite additive periodic components (cf. Section 8.4). Consequently, the cross correlation estimate

$$\hat{R}_{AB}(\tau)_T = \sum_n A_n \frac{1}{T} \int_{-T/2}^{T/2} \cos(\omega_n [t+\tau] + \Theta_n) B(t) dt$$

and $\hat{R}_{BA}(\tau)_T$ must converge to zero. Thus, we have

$$\lim_{T \to \infty} \hat{R}_X(\tau)_T = \lim_{T \to \infty} \hat{R}_A(\tau)_T + R_B(\tau).$$

Also, we have

$$R_X(\tau) = R_A(\tau) + R_B(\tau)$$

since $R_{AB}(\tau) \equiv R_{BA}(\tau) \equiv 0$. Hence, $X(t)$ has mean-square ergodicity of the autocorrelation if and only if

$$\lim_{T \to \infty} \hat{R}_A(\tau)_T = R_A(\tau),$$

which requires that $A(t)$ have mean-square ergodicity of the autocorrelation. But, we can show that

$$\lim_{T \to \infty} \hat{R}_A(\tau) = \frac{1}{2} \sum_n A_n^2 \cos(\omega_n \tau)$$

and

$$R_A(\tau) = \frac{1}{2} \sum_n E\{A_n^2\} \cos(\omega_n \tau).$$

Therefore, it is required that $A_n^2 = E\{A_n^2\}$, which means that A_n^2 must be nonrandom.

8.11 Using the model $Y(t) = cx(t - \tau_*) + N(t)$, we obtain

$$\hat{R}_{xY}(\tau)_T \triangleq \frac{1}{T} \int_{-T/2}^{T/2} Y(t+\tau) x(t) dt = \frac{1}{T} \int_{-T/2}^{T/2} cx(t+\tau-\tau_*) x(t) dt + \frac{1}{T} \int_{-T/2}^{T/2} N(t+\tau) x(t) dt.$$

The mean of $\hat{R}_{xY}(\tau)_T$ is given by

$$\begin{aligned} E\{\hat{R}_{xY}(\tau)_T\} &= \frac{1}{T} \int_{-T/2}^{T/2} cx(t+\tau-\tau_*) x(t) dt + \frac{1}{T} \int_{-T/2}^{T/2} E\{N(t+\tau)\} x(t) dt \\ &= \frac{1}{T} \int_{-T/2}^{T/2} cx(t+\tau-\tau_*) x(t) dt = c\hat{R}_x(\tau-\tau_*)_T, \end{aligned}$$

which tends to peak at $\tau = \tau_*$, especially for large T. The variance of $\hat{R}_{xY}(\tau)_T$ is given by

$$\begin{aligned} Var\{\hat{R}_{xY}(\tau)_T\} &= E\{[\hat{R}_{xY}(\tau)_T - E\{\hat{R}_{xY}(\tau)_T\}]^2\} \\ &= \frac{1}{T^2} \iint_{-T/2}^{T/2} E\{N(t+\tau) N(u+\tau)\} x(t) x(u) dt du \end{aligned}$$

$$= \frac{1}{T^2} \iint_{-T/2}^{T/2} R_N(t-u)x(t)x(u)dtdu$$

$$= \frac{1}{T^2} \iint_{-T/2}^{T/2} N_0\delta(t-u)x(t)x(u)dtdu = \frac{N_0}{T^2} \int_{-T/2}^{T/2} x^2(t)dt \propto \frac{1}{T},$$

which decreases with increasing T. Thus, for large enough integration time T, $\hat{R}_{xY}(\tau)_T$ will reach its peak at τ very close to τ_*.

8.12 *a*) The output of the appropriate device is given by

$$\hat{R}_X(\tau_*) = \int_{-\infty}^{\infty} h(t-u)X(u)X(u-\tau_*)du.$$

b) If the lowpass filter has impulse-response function

$$h(t-u) = \begin{cases} 1/T, & t-T \le u \le t \\ 0, & \textit{otherwise}, \end{cases}$$

then $\hat{R}_X(\tau_*)$ in part *a* becomes

$$\hat{R}_X(\tau_*)_T = \frac{1}{T} \int_{t-T}^{t} X(u)X(u-\tau_*)du.$$

c) The mean of $\hat{R}_X(\tau_*)_T$ is given by

$$Mean = E\{\hat{R}_X(\tau_*)_T\} = \frac{1}{T} \int_{t-T}^{t} E\{X(u)X(u-\tau_*)\}du = \frac{1}{T} \int_{t-T}^{t} K_X(\tau_*)du = K_X(\tau_*).$$

The variance of $\hat{R}_X(\tau_*)_T$ is given by

$$Var = E\{[\hat{R}_X(\tau_*)_T]^2\} - (Mean)^2.$$

Using Isserlis' formula to evaluate the first term yields

$$E\{[\hat{R}_X(\tau_*)_T]^2\} = \frac{1}{T^2} \iint_{t-T}^{t} E\{X(u)X(u-\tau_*)X(v)X(v-\tau_*)\}dudv$$

$$= \frac{1}{T^2} \iint_{t-T}^{t} [K_X^2(\tau_*) + K_X^2(u-v) + K_X(u-v+\tau_*)K_X(u-v-\tau_*)]dudv$$

$$= \frac{1}{T^2} \iint_{0}^{T} [K_X^2(\tau_*) + K_X^2(u-v) + K_X(u-v+\tau_*)K_X(u-v-\tau_*)]dudv$$

$$= K_X^2(\tau_*) + \frac{2}{T} \int_0^T (1-\frac{w}{T})K_X^2(w)dw + \frac{2}{T} \int_0^T (1-\frac{w}{T})K_X(w+\tau_*)K_X(w-\tau_*)dw$$

$$= K_X^2(\tau_*) + \frac{\sigma^4\tau_0}{T}[1 - \frac{\tau_0}{2T}(1 - e^{-2T/\tau_0})]$$

$$+ \frac{2\sigma^4}{T}[e^{-2\tau_*/\tau_0}(\tau_* - \frac{\tau_*^2}{2T} - \frac{\tau_0\tau_*}{2T} + \frac{\tau_0}{2} + \frac{1}{T}(\frac{\tau_0}{2})^2) - \frac{1}{T}(\frac{\tau_0}{2})^2 e^{-2T/\tau_0}].$$

Thus,

$$Var = \frac{\sigma^4\tau_0}{T}[1 - \frac{\tau_0}{2T}(1 - e^{-2T/\tau_0})]$$

$$+ 2\sigma^4 e^{-2\tau_*/\tau_0}[\frac{\tau_*}{T} - \frac{\tau_*^2}{2T^2} - \frac{\tau_0\tau_*}{2T^2} + \frac{\tau_0}{2T} + \frac{1}{T^2}(\frac{\tau_0}{2})^2] - \frac{\sigma^4}{2}(\frac{\tau_0}{T})^2 e^{-2T/\tau_0}.$$

When $T \gg \tau_0$, we have

$$Var \simeq \frac{2\tau_*}{T}(1 - \frac{\tau_*}{2T})\sigma^4 e^{-2\tau_*/\tau_0},$$

which is much smaller than

$$(Mean)^2 = \sigma^4 e^{-2\tau_*/\tau_0},$$

when $T \gg \tau_*$.

d) According to the figure, we have

$$Y(t) = \int_{t-T}^{t} [X(u) + X(u - \tau_*)]^2 du.$$

The mean of $Y(t)$ is given by

$$E\{Y(t)\} = \int_{t-T}^{t} E\{[X(u) + X(u - \tau_*)]^2\}du = \int_{t-T}^{t} [R_X(0) + R_X(0) + 2R_X(\tau_*)]du$$

$$= 2[R_X(0) + R_X(\tau_*)].$$

8.13 The time-average mean of $X(t)$ is given by

$$\hat{m}_X(T) = \frac{1}{T}\int_{-T/2}^{T/2} X(t)dt$$

and $E\{\hat{m}_X(T)\} = m_X$. The variance of $\hat{m}_X(T)$ is given by

$$Var = E\{\hat{m}_X^2(T)\} - m_X^2 = \frac{1}{T^2}\iint_{-T/2}^{T/2} E\{X(t)X(u)\}dtdu - m_X^2$$

$$= \frac{1}{T}\int_{-T}^{T}(1 - \frac{|\tau|}{T})R_X(\tau)d\tau - m_X^2 = \frac{2}{T}\int_{0}^{T}(1 - \frac{\tau}{T})K_X(\tau)d\tau.$$

Since $K_X(\tau) \ll K_X(0)$ for $|\tau| > \tau_0$, then if $T \gg \tau_0$, it follows that

$$Var \simeq \frac{2}{T}\int_0^T K_X(\tau)d\tau \simeq \frac{K_X(0)}{T/\tau_0}.$$

From the specification $Var \le \frac{1}{100}K_X(0)$, we obtain an approximate required averaging time of $T \ge 100\tau_0$. Since the effective number of uncorrelated time samples in an interval of length of T is $N = T/\tau_0$ (cf. (8.28)) and since the variance of the estimate that uses N uncorrelated samples is $K_X(0)/N$ (cf. (8.21)), then $N = 100$ is approximately the minimum number of samples that will result in a variance as small as $K_X(0)/100$.

8.14 Since $X(t)$ and $X(t+\tau)$ become statistically independent as $\tau \to \infty$, then

$$R_X(\tau) = E\{X(t+\tau)X(t)\} \quad \to \quad E\{X(t+\tau)\}E\{X(t)\} = m_X^2.$$

Therefore, $K_X(\tau) \to 0$ and, as a result, (8.42c) is satisfied. Consequently, (8.42b) is satisfied. Thus, $X(t)$ exhibits mean-square ergodicity of the mean. From (8.49) and the statistical independence of $X(t_1+u)X(t_2+u)$ and $X(t_1)X(t_2)$ as $u \to \infty$, we see that

$$\begin{aligned}\lim_{u\to\infty} K_{Y_\tau}(u) &= \lim_{u\to\infty} E\{X(t+\tau+u)X(t+u)X(t+\tau)X(t)\} - R_X^2(\tau)\\ &= \lim_{u\to\infty} E\{X(t+\tau+u)X(t+u)\}E\{X(t+\tau)X(t)\} - R_X^2(\tau)\\ &= R_X(\tau)R_X(\tau) - R_X^2(\tau) = 0.\end{aligned}$$

Therefore, (8.48b) is satisfied and $X(t)$ exhibits mean-square ergodicity of the autocorrelation.

8.15 Let $\{\tau\} = \{\tau_1, \tau_2, \tau_3\}$ and

$$Z_{\{\tau\}}(t) = X(t)X(t+\tau_1)X(t+\tau_2)X(t+\tau_3)$$

and also

$$m_{Z_{\{\tau\}}} \triangleq E\{Z_{\{\tau\}}(t)\}, \qquad \hat{m}_{Z_{\{\tau\}}}(T) \triangleq \frac{1}{T}\int_{-T/2}^{T/2} Z_{\{\tau\}}(t)dt.$$

Then it follows from (8.42) that

$$\lim_{T\to\infty} E\{[m_{Z_{\{\tau\}}} - \hat{m}_{Z_{\{\tau\}}}(T)]^2\} = 0$$

if and only if

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T K_{Z_{\{\tau\}}}(u)du = 0,$$

where

$$K_{Z_{\{\tau\}}}(u) = E\{Z_{\{\tau\}}(t+u)Z_{\{\tau\}}(t)\} - E\{Z_{\{\tau\}}(t+u)\}E\{Z_{\{\tau\}}(t)\}.$$

For a stationary zero-mean Gaussian process $X(t)$, we can use Isserlis' formula (5.58) to obtain

$$\begin{aligned} E\{Z_{\{\tau\}}(t+u)Z_{\{\tau\}}(t)\} &= E\{X(t+\tau_1+u)X(t+\tau_2+u)X(t+\tau_3+u) \\ &\qquad \times X(t+u)X(t+\tau_1)X(t+\tau_2)X(t+\tau_3)X(t)\} \\ &= [K_X(\tau_1-\tau_2)K_X(\tau_3) + K_X(\tau_1-\tau_3)K_X(\tau_2) + K_X(\tau_1)K_X(\tau_2-\tau_3)]^2 \\ &\qquad + \sum K_X(u+v_1)K_X(u+v_2)K_X(u+v_3)K_X(u+v_4), \end{aligned}$$

where $\sum$ contains 96 terms of the form shown in which v_i, $i=1, \ldots, 4$ are zero or a sum of some or all of $\pm\tau_i$ for $i=1, 2, 3$. We can also use Isserlis' formula to obtain

$$\begin{aligned} &E\{Z_{\{\tau\}}(t+u)\}E\{Z_{\{\tau\}}(t)\} \\ &\qquad = [K_X(\tau_1-\tau_2)K_X(\tau_3) + K_X(\tau_1-\tau_3)K_X(\tau_2) + K_X(\tau_1)K_X(\tau_2-\tau_3)]^2. \end{aligned}$$

Hence, we have

$$K_{Z_{\{\tau\}}}(u) = \sum K_X(u+v_1)K_X(u+v_2)K_X(u+v_3)K_X(u+v_4).$$

Since $K_X(u) \to 0$ as $u \to \infty$, it follows from this equation that $K_{Z_{\{\tau\}}}(u) \to 0$ as $u \to \infty$ and, therefore,

$$\lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} K_{Z_{\{\tau\}}}(u)du = 0.$$

Thus, $X(t)$ exhibits mean-square ergodicity of the fourth moment.

8.16 By using Isserlis' formula (5.58), high-order moments of a stationary Gaussian process can be estimated by estimating only the second-order moments and substituting the estimates into (5.58) as if the estimates were the true second-order moments.

8.17 The empirical autocorrelation of $X(t)$ given in (8.84) is obtained as follows:

$$\begin{aligned} \hat{R}_X(\tau) &\triangleq \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} X(t+\tau)X(t)dt \\ &= \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} A(t+\tau)A(t)\frac{1}{2}\{\cos(\omega_0\tau) + \cos(\omega_0[2t+\tau]+2\Theta)\}dt \\ &= \frac{1}{2}\lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} A(t+\tau)A(t)dt\cos(\omega_0\tau) \end{aligned}$$

$$+ \frac{1}{2}\mathrm{Re}\{\lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} A(t+\tau)A(t)e^{-i2\omega_0 t}dte^{-i\omega_0\tau}e^{-i2\Theta}\}$$

$$= \frac{1}{2}\hat{R}_A(\tau)\cos(\omega_0\tau) + \frac{1}{2}\mathrm{Re}\{\hat{R}_A^{2\omega_0}(\tau)e^{-i\omega_0\tau}e^{-i2\Theta}\},$$

where $\hat{R}_A^{2\omega_0}(\tau)$ is defined in (8.87). The probabilistic autocorrelation for $X(t)$ is obtained as follows:

$$R_X(t+\tau, t) \triangleq E\{X(t+\tau)X(t)\}$$

$$= E\{A(t+\tau)A(t)\}\frac{1}{2}[\cos(\omega_0\tau) + E\{\cos(\omega_0[2t+\tau]+2\Theta)\}]$$

$$= \frac{1}{2}R_A(t+\tau, t)\cos(\omega_0\tau) + \frac{1}{2}\mathrm{Re}\{R_A(t+\tau, t)e^{-i2\omega_0 t}e^{-i\omega_0\tau}E\{e^{-i2\Theta}\}\},$$

where the statistical independence of $A(t)$ and Θ has been used. The time-averaged value of $R_X(t+\tau, t)$ can now be obtained as follow:

$$<R_X>(\tau) \triangleq \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} R_X(t+\tau, t)dt$$

$$= \frac{1}{2}\lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} R_A(t+\tau, t)dt\cos(\omega_0\tau)$$

$$+ \frac{1}{2}\mathrm{Re}\{\lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} R_A(t+\tau, t)e^{-i2\omega_0 t}dte^{-i\omega_0\tau}E\{e^{-i2\Theta}\}\}$$

$$= \frac{1}{2}E\{\lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} A(t+\tau)A(t)dt\}\cos(\omega_0\tau)$$

$$+ \frac{1}{2}\mathrm{Re}\{E\{\lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} A(t+\tau)A(t)e^{-i2\omega_0 t}dt\}e^{-i\omega_0\tau}E\{e^{-i2\Theta}\}\}$$

$$= \frac{1}{2}E\{\hat{R}_A(\tau)\}\cos(\omega_0\tau) + \frac{1}{2}\mathrm{Re}\{E\{\hat{R}_A^{2\omega_0}(\tau)\}e^{-i\omega_0\tau}E\{e^{-i2\Theta}\}\} = E\{\hat{R}_X(\tau)\}.$$

8.18 We first define

$$h(t) \triangleq \begin{cases} 1, & |t| \le T/2 \\ 0, & \textit{otherwise}. \end{cases}$$

Then (8.70) can be expressed as

$$\frac{1}{T^2}\iint_{-T/2}^{T/2} K_X(t, u)dtdu = \frac{1}{T^2}\iint_{-\infty}^{\infty} K_X(t, u)h(t)h(u)dtdu.$$

We now use the change of variables $\tau = t - u$ and then $w = u + \tau/2$ to obtain

$$\frac{1}{T^2}\iint_{-T/2}^{T/2} K_X(t, u)dtdu = \frac{1}{T^2}\iint_{-\infty}^{\infty} K_X(\tau+u, u)h(\tau+u)h(u)d\tau du$$

$$= \frac{1}{T^2}\iint_{-\infty}^{\infty} K_X(w+\tau/2, w-\tau/2)h(w+\tau/2)h(w-\tau/2)dwd\tau.$$

Since

$$h(w+\tau/2)h(w-\tau/2) = \begin{cases} 1, & |w| \leq (T-|\tau|)/2 \text{ and } |\tau| \leq T \\ 0, & otherwise, \end{cases}$$

then we obtain the equivalent expression (8.113). The limit (8.114) is not necessarily the same as (8.113) since the integrand in (8.113) approaches zero rather than $\langle K_X\rangle(\tau)$ as $|\tau| \to T$:

$$\int_{-(T-|\tau|)/2}^{(T-|\tau|)/2} K_X(t+\tau/2, t-\tau/2)dt \to 0 \quad \text{as} \quad |\tau| \to T.$$

However, if $\langle K_X\rangle(\tau) \to 0$ as $\tau \to \infty$, then (8.114) would be equivalent to (8.113), since both would equal zero. Thus, the analog (8.114) of (8.42c) is a sufficient condition for mean-square ergodicity of the mean of a regular nonstationary process.

8.19 For a Poisson process $N(t)$, we have

$$E\{N(t)\} = Var\{N(t)\} = \lambda t.$$

For $\hat{\lambda}(T) = N(T)/T$, we have

$$E\{\hat{\lambda}(T)\} = E\{N(T)\}/T = \lambda,$$

$$Var\{\hat{\lambda}(T)\} = E\{[\frac{N(T)}{T} - \lambda]^2\} = \frac{1}{T^2}E\{[N(T)-\lambda T]^2\}$$

$$= \frac{1}{T^2}Var\{N(T)\} = \frac{\lambda}{T}.$$

Therefore, we obtain

$$\lim_{T\to\infty} E\{[\hat{\lambda}(T)-\lambda]^2\} = \lim_{T\to\infty} [E\{\hat{\lambda}^2(T)\} - 2\lambda E\{\hat{\lambda}(T)\} + \lambda^2]$$

$$= \lim_{T\to\infty} [\frac{\lambda}{T} + \lambda^2 - 2\lambda\lambda + \lambda^2] = 0.$$

8.20 Let $W(t) \triangleq u[x - X(t)]$. Then

$$\hat{m}_W(T) = \frac{1}{T}\int_0^T u[x - X(t)]dt = \hat{F}_X(x)_T$$

is the finite-time fraction-of-time distribution, and

$$m_W = E\{u[x - X(t)]\} = F_X(x)$$

is the probabilistic distribution. It follows that

$$\lim_{T\to\infty} E\{[\hat{F}_X(x)_T - F_X(x)]^2\} = 0$$

if and only if

$$\lim_{T\to\infty} E\{[\hat{m}_W(T) - m_W]^2\} = 0,$$

which holds if and only if (cf.(8.42))

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T K_W(\tau)d\tau = 0,$$

where

$$\begin{aligned} K_W(\tau) &= E\{W(t+\tau)W(t)\} - [E\{W(t)\}]^2 \\ &= E\{u[x - X(t+\tau)]u[x - X(t)]\} - F_X^2(x) = F_{X(t+\tau)X(t)}(x, x) - F_X^2(x). \end{aligned}$$

8.21 The necessary and sufficient condition for mean-square ergodicity of the distribution is, from exercise 8.20,

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T K_W(\tau)d\tau = 0,$$

where

$$K_W(\tau) \triangleq F_{X(t+\tau)X(t)}(x, x) - F_X^2(x).$$

Thus, it is required that

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T F_{X(t+\tau)X(t)}(x, x)d\tau = F_X^2(x). \qquad (*)$$

Since we are given that $X(t+\tau)$ and $X(t)$ become statistically independent as $\tau \to \infty$, then

$$\lim_{\tau\to\infty} F_{X(t+\tau)X(t)}(x, x) = F_X^2(x)$$

and, as a result, (*) is satisfied.

8.22 Since $F^1_{X(t_1)\cdots X(t_n)}$ and $F^2_{X(t_1)\cdots X(t_n)}$ are distinct for $X(t)$ given by

$$X(t) = \begin{cases} X_1(t), & \textit{probability} = p > 0 \\ X_2(t), & \textit{probability} = 1 - p > 0, \end{cases}$$

then some of their moments must be distinct. Let us assume that the first moments are distinct:

$$m_{X_1} = E\{X_1(t)\} \neq m_{X_2} = E\{X_2(t)\}.$$

Since $X_1(t)$ and $X_2(t)$ each exhibit ergodicity of the mean, then the empirical mean of $X(t)$ is given by

$$<X(t)> = \begin{cases} <X_1(t)> = m_{X_1}, & \textit{probability} = p > 0 \\ <X_2(t)> = m_{X_2}, & \textit{probability} = 1-p > 0. \end{cases}$$

However, the probabilistic mean of $X(t)$ is

$$E\{X(t)\} = pm_{X_1} + (1-p)m_{X_2}.$$

Thus, in general $E\{X(t)\} \neq <X(t)>$ and $X(t)$, therefore, does not exhibit mean-square ergodicity of the mean.

8.23 *a*) From (8.97), we have

$$\hat{F}_X(y) \triangleq \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} u[y - X(t)]dt.$$

Using (8.100) and interchanging the order of the time-averaging and differentiation operations yields

$$\hat{f}_X(y) = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} \delta[y - X(t)]dt, \tag{*}$$

since

$$\frac{du(y)}{dy} = \delta(y).$$

Substituting (*) into (8.99) and interchanging the order of the time-averaging and integration operations yields

$$\hat{m}_X = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} \int_{-\infty}^{\infty} y\,\delta[y - X(t)]dydt = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} x(t) \int_{-\infty}^{\infty} \delta[y - X(t)]dydt$$

$$= \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} X(t)dt,$$

which is (8.95), as desired.

b) From (8.101), we have

$$\hat{F}_{X(t_1)X(t_2)}(y_1, y_2) \triangleq \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} u[y_1 - X(t_1+t)]u[y_2 - X(t_2+t)]dt.$$

Using (8.103) and interchanging the order of the time-averaging and differentiation operations yields

$$\hat{f}_{X(t_1)X(t_2)}(y_1, y_2) = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} \frac{\partial}{\partial y_1} u[y_1 - X(t_1+t)] \frac{\partial}{\partial y_2} u[y_2 - X(t_2+t)] dt$$

$$= \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} \delta[y_1 - X(t_1+t)]\delta[y_2 - X(t_2+t)] dt. \qquad (**)$$

Substituting (**) into (8.102) and interchanging the order of time-averaging and integration operations yields

$$\hat{R}_X(t_1 - t_2) = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} \int_{-\infty}^{\infty} y_1 \delta[y_1 - X(t_1+t)] dy_1 \int_{-\infty}^{\infty} y_2 \delta[y_2 - X(t_2+t)] dy_2 \, dt$$

$$= \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} X(t_1+t) X(t_2+t) dt.$$

Using the change of variables $t = s - t_2$ yields

$$\hat{R}_X(t_1 - t_2) = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2+t_2}^{T/2+t_2} X(s+t_1-t_2) X(s) ds = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} X(s+t_1-t_2) X(s) ds,$$

which is (8.96), as desired.

c) From (8.100), we have

$$\hat{f}_X(y) = \frac{d\hat{F}_X(y)}{dy} = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} \delta[y - x(t)] dt \geq 0$$

since $\delta \geq 0$. Therefore, $\hat{F}_X(y)$ is a non-decreasing function of y. Also, from (8.98) we observe that

$$u[y - x(t)] = 0 \text{ for } y = -\infty \quad \text{and} \quad u[y - x(t)] = 1 \text{ for } y = \infty.$$

Therefore, we have

$$\hat{F}_X(-\infty) = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} 0 \, dt = 0,$$

$$\hat{F}_X(\infty) = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} 1 \, dt = 1.$$

Similarly, since

$$\frac{\partial^2}{\partial y_1 \partial y_2} \hat{F}_{X(t_1)X(t_2)}(y_1, y_2) = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} \delta[y_1 - x(t+t_1)]\delta[y_2 - x(t+t_2)] dt \geq 0,$$

$\hat{F}_{X(t_1)X(t_2)}(y_1, y_2)$ is a non-decreasing function of y_1 and y_2. Also,

$$\hat{F}_{X(t_1)X(t_2)}(-\infty, -\infty) = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} 0 \times 0 \, dt = 0$$

$$\hat{F}_{X(t_1)X(t_2)}(\infty, \infty) = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} 1 \times 1 \, dt = 1.$$

Thus, $\hat{F}_{X(t)}(y)$ and $\hat{F}_{X(t_1)X(t_2)}(y_1, y_2)$ are valid distribution functions.

Chapter 9

Linear Transformations, Filters, and Dynamical Systems

9.1 Using the results from (9.12) and (9.13) yields

$$\mathbf{K_Y} \triangleq E\{(\mathbf{Y}-\mathbf{m_Y})(\mathbf{Y}-\mathbf{m_Y})^T\} = \mathbf{R_Y} - \mathbf{m_Y}\mathbf{m_Y^T}$$
$$= \mathbf{HR_XH}^T - \mathbf{Hm_X}(\mathbf{Hm_X})^T = \mathbf{H}(\mathbf{R_x}-\mathbf{m_X}\mathbf{m_X^T})\mathbf{H}^T = \mathbf{HK_XH}^T.$$

9.2 *a*) Substituting the input

$$X(i) = \delta_{i-k} = \begin{cases} 1, & i = k \\ 0, & otherwise \end{cases}$$

into (9.15) yields the desired result,

$$Y(i) = \sum_{i=-\infty}^{\infty} h(i, j)X(j) = \sum_{j=-\infty}^{\infty} h(i, j)\delta_{j-k} = h(i, k).$$

b) Letting $k = i - j$ in (9.18), we obtain

$$Y(i) = \sum_{j=-\infty}^{\infty} h(i-j)X(j) = \sum_{k=\infty}^{-\infty} h(k)X(i-k) = \sum_{k=-\infty}^{\infty} h(k)X(i-k).$$

9.3 For a WSS process $X(t)$, we have $m_X(i) = m_X$ and $R_X(i, j) = R_X(i-j)$. Therefore, (9.23) yields

$$m_Y(i) = m_X(i) \otimes h(i) = \sum_{j=-\infty}^{\infty} m_X(i-j)h(j) = m_X \sum_{j=-\infty}^{\infty} h(j),$$

which is (9.25). Also, (9.24) yields

$$R_Y(i, j) = h(i) \otimes R_X(i, j) \otimes h(j) = \sum_{l,k=-\infty}^{\infty} R_X(i-j+l-k)h(k)h(l).$$

Using the change of variables $k - l = r$ in this equation results in

$$R_Y(i, j) = \sum_{r,l=-\infty}^{\infty} R_X(i-j-r)h(l+r)h(l) = \sum_{r=-\infty}^{\infty} R_X(i-j-r)[h(r) \otimes h(-r)]$$
$$= [R_X(k) \otimes r_h(k)]_{k=i-j} = R_Y(i-j),$$

where $r_h(k)$ is given by (9.27).

9.4 Since $X(i)$ and $Y(i)$ are related by a causal linear time-invariant filter h and are WSS, then the two cross-correlations are given by

$$
\begin{aligned}
R_{YX}(k) &= E\{Y(i+k)X(i)\} = E\{\sum_{l=0}^{\infty} h(l)X(i+k-l)X(i)\} \\
&= \sum_{l=0}^{\infty} h(l)E\{X(i+k-l)X(i)\} = \sum_{l=0}^{\infty} h(l)R_X(k-l) \\
&= \sum_{l=-\infty}^{\infty} h(l)R_X(k-l) = R_X(k) \otimes h(k)
\end{aligned}
$$

and

$$
\begin{aligned}
R_{XY}(k) &= E\{X(i+k)Y(i)\} = E\{X(i+k)\sum_{l=0}^{\infty} h(l)X(i-l)\} \\
&= \sum_{l=0}^{\infty} h(l)E\{X(i+k)X(i-l)\} = \sum_{l=0}^{\infty} h(l)R_X(k+l) \\
&= \sum_{l=-\infty}^{0} h(-l)R_X(k-l) = \sum_{l=-\infty}^{\infty} h(-l)R_X(k-l) = R_X(k) \otimes h(-k).
\end{aligned}
$$

9.5 Since $r_h(k)$ in (9.27) satisfies

$$
\begin{aligned}
r_h(-k) &= \sum_{i=-\infty}^{\infty} h(i-k)h(i) = \sum_{j=-\infty}^{\infty} h(j)h(j+k) \qquad (\text{using } i-k=j) \\
&= r_h(k),
\end{aligned}
$$

then it is an even function and we therefore need to evaluate (9.27) for only $k \geq 0$. Using (9.35) in (9.27) yields

$$
\begin{aligned}
r_h(k) &= \sum_{i=-\infty}^{\infty} h(i+k)h(i) = \sum_{i=1}^{\infty} h(i+k)h(i) \\
&= \sum_{i=1}^{\infty} \frac{1}{a^2} a^{i+k} a^i = a^{k-2} \sum_{i=1}^{\infty} a^{2i} = a^{k-2} \frac{a^2}{1-a^2} \qquad (\text{using } (9.36)) \\
&= \frac{a^k}{1-a^2}, \qquad k \geq 0,
\end{aligned}
$$

or, equivalently,

$$
r_h(k) = \frac{a^{|k|}}{1-a^2}, \qquad -\infty < k < \infty,
$$

which is the desired result (9.38).

9.6 *a*) Substituting $X(t) = \delta(t-v)$ into (9.41) yields the output waveform

$$Y(t) = \int_{-\infty}^{\infty} h(t, u)X(u)du = \int_{-\infty}^{\infty} h(t, u)\delta(u-v)du = h(t, v).$$

b) For a time-invariant system, we have $h(t, u) = h(t-u)$. Therefore, from (9.43) with $R_X(r, s) = R_X(r-s)$, we obtain

$$R_Y(t, u) = \iint_{-\infty}^{\infty} h(t-r)h(u-s)R_X(r-s)drds.$$

Using the change of variables $t - r = z + v$ and $u - s = z$ in this equation yields

$$R_Y(t, u) = \iint_{-\infty}^{\infty} h(z+v)h(z)R_X(t-u-v)dzdv = \int_{-\infty}^{\infty} R_X(t-u-v)[h(v) \otimes h(-v)]dv$$

$$= [R_X(\tau) \otimes r_h(\tau)]_{\tau=t-u} = R_Y(t-u),$$

where $r_h(\tau)$ is given by (9.46).

c) For a time-invariant system, (9.41) becomes

$$Y(t) = \int_{-\infty}^{\infty} h(t-u)X(u)du$$

and the two cross-correlations of the input and output are then given by

$$R_{YX}(\tau) = E\{Y(t+\tau)X(t)\} = E\{\int_{-\infty}^{\infty} h(t+\tau-u)X(t)X(u)du\}$$

$$= \int_{-\infty}^{\infty} h(\tau-u+t)R_X(u-t)du = \int_{-\infty}^{\infty} h(\tau-v)R_X(v)dv \quad (\text{using } v = u - t)$$

$$= R_X(\tau) \otimes h(\tau)$$

and

$$R_{XY}(\tau) = E\{X(t+\tau)Y(t)\} = E\{\int_{-\infty}^{\infty} h(t-u)X(t+\tau)X(u)du\}$$

$$= \int_{-\infty}^{\infty} h(t-u)R_X(t-u+\tau)du = \int_{-\infty}^{\infty} h(-v)R_X(\tau-v)dv \quad (\text{using } v = u - t)$$

$$= R_X(\tau) \otimes h(-\tau).$$

9.7 Using the definition (3.2) of the empirical autocorrelation, we obtain

$$\hat{R}_h(\tau) = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} h(t+\tau)h(t)dt$$

$$= \sum_{n,m} h_n h_m \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} e^{in\omega_0(t+\tau)} e^{im\omega_0 t} dt$$

$$= \sum_{n,m} h_n h_m e^{in\omega_0\tau} \lim_{T\to\infty} \frac{\sin([n+m]\omega_0 T/2)}{[n+m]\omega_0 T/2} = \sum_n |h_n|^2 e^{in\omega_0\tau},$$

since

$$\lim_{T\to\infty} \frac{\sin(q\,\omega_0 T/2)}{q\,\omega_0 T/2} = \delta_q \quad \text{and} \quad h_{-m} = h_m^*.$$

9.8 The mean and covariance of $Z(t)$ are given by

$$m_Z(t) = m_Z\lambda, \quad t \ge 0$$

$$K_Z(t_1, t_2) = \lambda\sigma_Z^2\delta(t_1 - t_2), \quad t_1, t_2 \ge 0.$$

Therefore,

$$m_X(t) = E\{X(t)\} = E\{h(t) \otimes Z(t)\}$$

$$= E\{\int_{-\infty}^{\infty} h(t-u)Z(u)du\} = \int_0^{\infty} h(t-u)m_Z(u)du = m_Z\lambda \int_{-\infty}^{t} h(v)dv,$$

from which we obtain

$$\lim_{t\to\infty} m_X(t) = m_Z\lambda \int_{-\infty}^{\infty} h(v)dv.$$

Also,

$$R_X(t+\tau, t) = E\{X(t+\tau)X(t)\} = E\{[h(t+\tau) \otimes Z(t+\tau)][h(t) \otimes Z(t)]\}$$

$$= \iint_{-\infty}^{\infty} h(t+\tau-u)h(t-v)E\{Z(u)Z(v)\}dudv$$

$$= \iint_{0}^{\infty} h(t+\tau-u)h(t-v)[K_Z(u, v) + m_Z(u)m_Z(v)]dudv$$

$$= \lambda\sigma_Z^2 \iint_{-\infty}^{t} h(\tau+u)h(v)\delta(u-v)dudv + (m_Z\lambda)^2 \int_{-\infty}^{t+\tau} h(u)du \int_{-\infty}^{t} h(v)dv,$$

from which we obtain

$$\lim_{t\to\infty} R_X(t+\tau, t) = \lambda\sigma_Z^2 r_h(\tau) + [m_Z\lambda \int_{-\infty}^{\infty} h(v)dv]^2.$$

Therefore,

$$\lim_{t\to\infty} K_X(t+\tau, t) = \lim_{t\to\infty} [R_X(t+\tau, t) - m_X(t+\tau)m_X(t)] = \lambda\sigma_Z^2 r_h(\tau).$$

9.9 Substituting (9.57) into (9.46) for $\tau > 0$ yields

$$r_h(\tau) = \int_{-\infty}^{\infty} h(t+\tau)h(t)dt = \int_0^{\infty} \beta^2 e^{-\gamma(t+\tau)}e^{-\gamma t}dt = \beta^2 e^{-\gamma\tau}\int_0^{\infty} e^{-\gamma 2t}dt = \frac{\beta^2}{2\gamma}e^{-\gamma\tau}, \quad \tau > 0.$$

Since $r_h(\tau)$ is even (cf. solution to exercise 9.5), then we have the desired result (9.58). This result is analogous to (9.38) for the discrete-time system.

9.10 In this exercise, we need to interchange limiting and infinite summation operations of the following type:

$$Q \triangleq \lim_{N\to\infty} \frac{1}{2N+1} \sum_{i=-N}^{N} \sum_{k=-\infty}^{\infty} a(i-k)b(k).$$

If $b(k)$ has finite support (or if $b(k)$ is absolutely summable or square summable (has finite energy) and $a(i)$ is a persistent (e.g., finite-average-power) sequence that satisfies mild restrictions), then

$$Q = \sum_{k=-\infty}^{\infty} b(k) \lim_{N\to\infty} \frac{1}{2N+1} \sum_{i=-N}^{N} a(i+k) \qquad (*)$$

provided that the limit in (*) exists. But if $a(k)$ and $b(k)$ are interchanged, then (*) is not valid. In fact (*) yields $Q = 0$ in this case. This arises in part *a* where $b(k)$ is a unit-pulse-response, or a product thereof, and $a(k)$ is a sample sequence of a WSS process, or a product thereof. Because of this, the desired results can be obtained in this exercise only by starting with

$$Y(i) = \sum_{k=-\infty}^{\infty} h(k)X(i-k),$$

not with

$$Y(i) = \sum_{j=-\infty}^{\infty} X(j)h(i-j).$$

The analogous situation for limiting and infinite integration operations arises in part *b* and dictates that we start with

$$Y(t) = \int_{-\infty}^{\infty} h(u)X(t-u)du,$$

not with

$$Y(t) = \int_{-\infty}^{\infty} h(t-v)X(v)dv.$$

a) The input-output relation for a discrete-time filter is

$$Y(i) = \sum_{k=-\infty}^{\infty} h(k)X(i-k).$$

The empirical mean of $Y(i)$ is defined by

$$\hat{m}_Y \triangleq \lim_{N\to\infty} \frac{1}{2N+1} \sum_{i=-N}^{N} Y(i) = \sum_{k=-\infty}^{\infty} \lim_{N\to\infty} \frac{1}{2N+1} \sum_{i=-N}^{N} h(k)X(i-k)$$

$$= \sum_{k=-\infty}^{\infty} h(k)\hat{m}_X = \hat{m}_X \sum_{k=-\infty}^{\infty} h(k),$$

which is (9.65). The empirical autocorrelation of $Y(i)$ is defined by

$$\hat{R}_Y(k) \triangleq \lim_{N\to\infty} \frac{1}{2N+1} \sum_{i=-N}^{N} Y(i+k)Y(i)$$

$$= \lim_{N\to\infty} \frac{1}{2N+1} \sum_{i=-N}^{N} \sum_{j=-\infty}^{\infty} h(l)h(j)X(i+k-l)X(i-j)$$

$$= \sum_{l,j=-\infty}^{\infty} h(l)h(j)\hat{R}_X(k+j-l).$$

Using the change of variables $r = l - j$ in this equation yields

$$\hat{R}_Y(k) = \sum_{j,r=-\infty}^{\infty} h(j+r)h(j)\hat{R}_X(k-r)$$

$$= \sum_{r=-\infty}^{\infty} [h(r) \otimes h(-r)]\hat{R}_X(k-r) = \hat{R}_X(k) \otimes r_h(k),$$

which is (9.66).

b) For continuous-time processes, the input and output of a filter are related by

$$Y(t) = \int_{-\infty}^{\infty} h(u)X(t-u)du.$$

The empirical mean of $Y(t)$ is given by

$$\hat{m}_Y = \langle Y(t)\rangle = \langle \int_{-\infty}^{\infty} h(u)X(t-u)du \rangle$$

$$= \int_{-\infty}^{\infty} h(u)\langle X(t-u)\rangle du = \int_{-\infty}^{\infty} h(u)\hat{m}_X du = \hat{m}_X \int_{-\infty}^{\infty} h(u)du,$$

which is the desired result (9.67). The empirical autocorrelation of $Y(t)$ is given by

$$\hat{R}_Y(\tau) = \langle Y(t+\tau)Y(t)\rangle = \langle \iint_{-\infty}^{\infty} h(u)h(v)X(t+\tau-u)X(t-v)dudv\rangle$$

$$= \iint_{-\infty}^{\infty} h(u)h(v) \langle X(t+\tau-u)X(t-v)\rangle \, dudv$$

$$= \iint_{-\infty}^{\infty} h(u)h(v)\hat{R}_X(\tau+v-u)dudv$$

$$= \iint_{-\infty}^{\infty} h(v+\sigma)h(v)\hat{R}_X(\tau-\sigma)dvd\sigma \quad (\text{using } u-v=\sigma)$$

$$= \int_{-\infty}^{\infty} r_h(\sigma)\hat{R}_X(\tau-\sigma)d\sigma = \hat{R}_X(\tau)\otimes r_h(\tau),$$

which is the desired result (9.68).

c) For an asymptotically mean stationary process, the autocorrelation of $Y(t)$ is given by

$$R_Y(t+\tau,\, t) = E\{Y(t+\tau)Y(t)\} = \iint_{-\infty}^{\infty} h(u)h(v)E\{X(t+\tau-u)X(t-v)\}dudv$$

$$= \iint_{-\infty}^{\infty} h(u)h(v)R_X(t+\tau-u,\, t-v)dudv.$$

Therefore, the time-averaged autocorrelation is given by

$$\langle R_Y\rangle(\tau) = \langle R_Y(t+\tau,\, t)\rangle = \iint_{-\infty}^{\infty} h(u)h(v) \langle R_X(t+\tau-u,\, t-v)\rangle \, dudv$$

$$= \iint_{-\infty}^{\infty} h(u)h(v)\langle R_X\rangle(\tau+v-u)\, dudv = \langle R_X\rangle(\tau)\otimes r_h(\tau)$$

as in part *b*. Similarly, we can show that

$$\langle m_Y\rangle = \langle m_X\rangle \int_{-\infty}^{\infty} h(t)dt.$$

9.11 *a*) The responses $Y_1(t)$ and $Y_2(t)$ are related to the excitations $X_1(t)$ and $X_2(t)$ by

$$Y_1(t) = \int_{-\infty}^{\infty} h_1(u)X_1(t-u)du \quad \text{and} \quad Y_2(t) = \int_{-\infty}^{\infty} h_2(v)X_2(t-v)dv.$$

For jointly WSS processes $X_1(t)$ and $X_2(t)$, $Y_1(t)$ and $Y_2(t)$ are also jointly WSS with cross-correlation given by

$$R_{Y_1Y_2}(\tau) = E\{Y_1(t+\tau)Y_2(t)\} = E\{\iint_{-\infty}^{\infty} h_1(u)h_2(v)X_1(t+\tau-u)X_2(t-v)dudv\}$$

$$= \iint_{-\infty}^{\infty} h_1(u)h_2(v)R_{X_1X_2}(\tau+v-u)dudv$$

$$= \iint_{-\infty}^{\infty} h_1(v+\sigma)h_2(v)R_{X_1X_2}(\tau-\sigma)dvd\sigma \quad (\text{using } u-v=\sigma)$$

$$= \int_{-\infty}^{\infty} r_{h_1h_2}(\sigma)R_{X_1X_2}(\tau-\sigma)d\sigma = R_{X_1X_2}(\tau)\otimes r_{h_1h_2}(\tau),$$

where

$$r_{h_1h_2}(\tau) = \int_{-\infty}^{\infty} h_1(t+\sigma)h_2(t)dt.$$

b) The excitations and responses $X(t)$, $W(t)$, and $Y(t)$ are related by

$$W(t) = \int_{-\infty}^{\infty} h_1(u)X(t-u)du \quad \text{and} \quad Y(t) = \int_{-\infty}^{\infty} h_2(v)W(t-v)dv.$$

Therefore, for a WSS process $X(t)$, $W(t)$ and $Y(t)$ are jointly WSS, and the cross-correlation of $W(t)$ and $Y(t)$ is given by

$$R_{WY}(\tau) = E\{W(t+\tau)Y(t)\} = E\{\int_{-\infty}^{\infty} h_2(v)W(t+\tau)W(t-v)dv\}$$

$$= \int_{-\infty}^{\infty} h_2(v)R_W(\tau+v)dv = R_W(\tau)\otimes h_2(-\tau)$$

$$= R_X(\tau)\otimes r_{h_1}(\tau)\otimes h_2(-\tau) \quad (\text{using } (9.45)).$$

9.12 Since H is statistically independent of X, then

$$R_Y(\tau) = E\{Y(t+\tau)Y(t)\} = E\{\iint_{-\infty}^{\infty} H(t+\tau-u)X(u)H(t-v)X(v)dudv\}$$

$$= \iint_{-\infty}^{\infty} E\{H(t+\tau-u)H(t-v)\}R_X(u-v)dudv.$$

Letting $t-v=w$ and $t-u=w-z$ results in

$$R_Y(\tau) = \iint_{-\infty}^{\infty} E\{H(\tau - z + w)H(w)\}R_X(z)dwdz$$

$$= \int_{-\infty}^{\infty} E\{\int_{-\infty}^{\infty} H(\tau - z + w)H(w)dw\}R_X(z)dz$$

$$= \int_{-\infty}^{\infty} E\{r_H(\tau - z)\}R_X(z)dz = R_X(\tau) \otimes E\{r_H(\tau)\}.$$

9.13 The input and output of the system are related by

$$Y(t) = h(t) \otimes [X(t) + N(t)] = \int_{-\infty}^{\infty} h(t - u)[X(u) + N(u)]du.$$

Using the assumption that $N(t)$ is orthogonal to $X(t)$, we obtain the following cross-correlation of $Y(t)$ and $N(t)$:

$$R_{YN}(\tau) = E\{Y(t + \tau)N(t)\} = E\{\int_{-\infty}^{\infty} h(t + \tau - u)[X(u)N(t) + N(u)N(t)]du\}$$

$$= \int_{-\infty}^{\infty} h(t + \tau - u)R_N(u - t)du = \alpha^2 \int_{-\infty}^{\infty} h(t + \tau - u)\delta(u - t)du = \alpha^2 h(\tau),$$

which is the desired result.

9.14 Since $|a| < 1$, then from (9.83) we have

$$\lim_{i \to \infty} R_Y(i, i) = \lim_{i \to \infty} [a^{2i} R_Y(0, 0) + \sigma^2 \frac{1 - a^{2i}}{1 - a^2}] = \frac{\sigma^2}{1 - a^2},$$

which is equivalent to $R_Y(0)$ in (9.40). Using (9.79) with $k = i - j$ fixed, we obtain

$$\lim_{i \to \infty} R_Y(i, j) = \lim_{i \to \infty} a^{i-j} R_Y(j, j) = \lim_{j \to \infty} a^k R_Y(j, j) = \sigma^2 \frac{a^k}{1 - a^2},$$

which is equivalent to (9.40).

9.15 *a*) With $j = 0$, $i \to i - 1$, and $\mathbf{X}(0) = \mathbf{0}$, (9.86) yields

$$\mathbf{X}(i) = \sum_{m=0}^{i-1} \mathbf{\Phi}(i, m + 1)\mathbf{b}(m)U(m).$$

Substituting this equation into (9.85b) with $d(i) \equiv 0$ yields

$$Y(i) = \mathbf{c}(i) \sum_{m=0}^{i-1} \mathbf{\Phi}(i, m + 1)\mathbf{b}(m)U(m) = \sum_{j=0}^{i-1} \mathbf{c}(i)\mathbf{\Phi}(i, j + 1)\mathbf{b}(j)U(j),$$

which reveals that, for the form (9.150), the unit-pulse response $h(i, j)$ is given by (9.151).

b) We first express (9.152) in the form of the state-variable recursion (9.85):

$$X(i+1) = A(i)X(i) + b(i)U(i), \qquad i \geq 0$$
$$Y(i) = c(i)X(i), \qquad i \geq 0,$$

where

$$A(i) = \frac{\alpha i}{i+1}, \quad b(i) = \beta, \quad c(i) = 1.$$

Then, we use (9.87) to obtain

$$\Phi(i, j) = \begin{cases} 1, & j = i \\ \frac{j}{i}\alpha^{i-j}, & j < i, \end{cases}$$

and we use (9.151) to obtain the unit-pulse response:

$$h(i, j) = c(i)\Phi(i, j+1)b(j) = \beta\frac{j+1}{i}\alpha^{i-j-1}, \quad 0 \leq j < i.$$

c) Applying expectation to (9.152) yields

$$m_Y(i+1) = \frac{\alpha i}{i+1}m_Y(i) + \beta m_U(i), \quad i \geq 0.$$

Using (9.152), we obtain for the variance of $Y(t)$, the equation

$$K_Y(i+1, i+1) = E\{[Y(i+1) - m_Y(i+1)]^2\}$$
$$= E\{[\frac{\alpha i}{i+1}[Y(i)-m_Y(i)] + \beta[U(i)-m_U(i)]]^2\}.$$

Then expanding the square and using (9.153) yields

$$K_Y(i+1, i+1) = (\frac{\alpha i}{i+1})^2 K_Y(i, i) + (\beta\sigma)^2, \quad i \geq 0. \qquad (*)$$

As an alternative, we can use the general solution (9.92). For this first-order system, we have $Y(i) \equiv \mathbf{X}(i)$, $\mathbf{A}(i) = \alpha i/(i+1)$, and $\mathbf{b}(i) = \beta$. Therefore, (9.92) yields the same result (*).

d) Using the result from *b*, we obtain the summation formula for the mean and variance of $Y(t)$,

$$m_Y(i) = E\{Y(i)\} = E\{\sum_{j=0}^{i-1} h(i, j)U(j)\} = E\{\beta\sum_{j=0}^{i-1} \alpha^{i-j-1}\frac{j+1}{i}U(j)\}$$
$$= \beta\sum_{j=0}^{i-1} \alpha^{i-j-1}\frac{j+1}{i}m_U(j), \quad i > 0$$

and

$$K_Y(i,i) = E\{[Y(i)-m_Y(i)]^2\}$$
$$= E\{\beta^2 \sum_{j,k=0}^{i-1} \alpha^{i-j-1}\alpha^{i-k-1}\frac{j+1}{i}\frac{k+1}{i}[U(j)-m_U(j)][U(k)-m_U(k)]\}$$
$$= \beta^2 \sum_{j,k=0}^{i-1} \alpha^{i-j-1}\alpha^{i-k-1}\frac{j+1}{i}\frac{k+1}{i}K_U(j,k)$$
$$= (\beta\sigma)^2 \sum_{j=0}^{i-1} (\alpha^{i-j-1}\frac{j+1}{i})^2, \quad i>0.$$

9.16 To obtain (9.90), we simply take the expected value of both sides of (9.85a) to obtain

$$\mathbf{m_X}(i+1) = E\{\mathbf{X}(i+1)\} = \mathbf{A}(i)E\{\mathbf{X}(i)\} + \mathbf{b}(i)E\{U(i)\}$$
$$= \mathbf{A}(i)\mathbf{m_X}(i) + \mathbf{b}(i)m_U(i), \quad i \geq 0.$$

To obtain (9.91), we simply multiply (9.86), with i replaced by $i-1$, by $U(j)$ and take the expected value of both sides to obtain

$$E\{\mathbf{X}(i)U(j)\} = \mathbf{\Phi}(i,j)E\{\mathbf{X}(j)U(j)\} + \sum_{m=j}^{i-1}\mathbf{\Phi}(i,m+1)\mathbf{b}(m)E\{U(m)U(j)\}, \quad i>j,$$

or, equivalently,

$$\mathbf{K}_{\mathbf{X}U}(i,j) = \mathbf{\Phi}(i,j)\mathbf{K}_{\mathbf{X}U}(j,j) + \sum_{m=j}^{i-1}\mathbf{\Phi}(i,m+1)\mathbf{b}(m)K_U(m,j), \quad i>j.$$

Now, using the conditions (9.88) and (9.89) in this equation yields

$$\mathbf{K}_{\mathbf{X}U}(i,j) = \sigma^2\mathbf{\Phi}(i,j+1)\mathbf{b}(j), \quad i>j,$$

which is the desired result (9.91).

By analogy with the method used to obtain (9.80) from (9.77), we can obtain (9.92) from (9.86), together with the assumptions (9.88) and (9.89), as follows:

$$\mathbf{K_X}(i+1,i+1) = E\{\mathbf{X}(i+1)\mathbf{X}^T(i+1)\}$$
$$= E\{[\mathbf{\Phi}(i+1,j)\mathbf{X}(j) + \sum_{m=j}^{i}\mathbf{\Phi}(i+1,m+1)\mathbf{b}(m)U(m)]$$
$$\times [\mathbf{\Phi}(i+1,j)\mathbf{X}(j) + \sum_{n=j}^{i}\mathbf{\Phi}(i+1,n+1)\mathbf{b}(n)U(n)]^T\},$$

since the cross terms are zero by virtue of (9.88). Thus,

$$\mathbf{K_X}(i+1,i+1) = \mathbf{\Phi}(i+1,j)\mathbf{K_X}(j,j)\mathbf{\Phi}^T(i+1,j)$$
$$+ \sum_{m=j}^{i}\mathbf{\Phi}(i+1,m+1)\mathbf{b}(m)\mathbf{K}_{\mathbf{X}U}^T(j,m)\mathbf{\Phi}^T(i+1,j)$$
$$+ \mathbf{\Phi}(i+1,j)\sum_{n=j}^{i}\mathbf{K}_{\mathbf{X}U}(j,n)\mathbf{b}^T(n)\mathbf{\Phi}^T(i+1,n+1)$$

$$+ \sum_{n=j}^{i} \sum_{m=j}^{i} \Phi(i+1, m+1)\mathbf{b}(m)K_U(m, n)\mathbf{b}^T(n)\Phi^T(i+1, n+1)$$

$$= \Phi(i+1, j)\mathbf{K_X}(j, j)\Phi^T(i+1, j)$$

$$+ \sigma^2 \sum_{n=j}^{i} \Phi(i+1, n+1)\mathbf{b}(n)\mathbf{b}^T(n)\Phi^T(i+1, n+1), \quad i \geq j \geq 0,$$

since the middle two terms are zero by virtue of (9.88) and since the last term simplifies with the use of (9.89). Letting $j = i$ and using (9.87a) now yields

$$\mathbf{K_X}(i+1, i+1) = \mathbf{A}(i)\mathbf{K_X}(i, i)\mathbf{A}^T(i) + \sigma^2\mathbf{b}(i)\mathbf{b}^T(i), \quad i \geq 0,$$

which is the desired result (9.92).

9.17 For a time-invariant system,

$$\Phi(i, j) = \Phi(i-j).$$

Therefore, (9.87a) yields

$$\Phi = \begin{cases} \mathbf{I}, & i = j \\ \mathbf{A}^{i-j}, & i > j. \end{cases}$$

From (9.87b), we obtain

$$\Phi(j-i) = (\mathbf{A}^{i-j})^{-1} = \mathbf{A}^{j-i}.$$

Putting these two results together yields the desired result (9.154)-(9.155).

9.18 *a*) Since $\mathbf{X}(t)$ is defined in (9.157) by

$$\mathbf{X}(i) \triangleq \begin{bmatrix} X_1(i) \\ X_2(i) \end{bmatrix} \quad \text{or} \quad \begin{aligned} X_1(i) &= Y(i) \\ X_2(i) &= Y(i+1) = X_1(i+1), \end{aligned}$$

then (9.156) can be expressed as

$$\begin{aligned} X_1(i+1) &= X_2(i) \\ X_2(i+1) &= -2X_2(i) - X_1(i) + 3U(i) \\ Y(i) &= X_1(i), \qquad i \geq 0 \end{aligned}$$

or, equivalently,

$$\begin{bmatrix} X_1(i+1) \\ X_2(i+1) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -1 \end{bmatrix} \begin{bmatrix} X_1(i) \\ X_2(i) \end{bmatrix} + \begin{bmatrix} 0 \\ 3 \end{bmatrix} U(i), \qquad i \geq 0$$

$$Y(i) = [\,1 \quad 0\,] \begin{bmatrix} X_1(i) \\ X_2(i) \end{bmatrix} \qquad i \geq 0,$$

which is of the form in (9.85) with $d = 0$ and **A**, **b**, and **c** given by (9.158).

b) Taking the expected value of both sides of (9.156) yields the following difference equation for the evolution of the mean,

$$m_Y(i+2) + 2m_Y(i+1) + m_Y(i) = 3m_U(i) = 3.$$

The difference equation for the evolution of the variance can be obtained as follows:

$$K_Y(i+2, i+2) = E\{[Y(i+2) - m_Y(i+2)]^2\}$$
$$= E\{(-2[Y(i+1) - m_Y(i+1)] - [Y(i) - m_Y(i)] + 3[U(i) - m_U(i)])^2\}$$
$$= 4K_Y(i+1, i+1) + K_Y(i, i) + 2K_Y(i+1, i) + 2K_Y(i, i+1) + 9\sigma_U^2$$

(using $K_{YU}(i, j) = 0$, $j \geq i-1$). Thus,

$$K_Y(i+2, i+2) = 4K_Y(i+1, i+1) + K_Y(i, i) + 4K_Y(i+1, i) + 36, \quad i \geq 0,$$

since $K_Y(i, j) = K_Y(j, i)$. As an alternative, we can obtain the same result by substituting (9.158) into the general solutions (9.92), (9.95), and (9.96).

9.19 Using (9.159) and (9.160) to differentiate $X(t)$ in (9.107) yields

$$\dot{X}(t) = X(v)\frac{d}{dt}\Phi(t, v) + \frac{d}{dt}\int_v^t \Phi(t, w)b(w)U(w)dw, \quad 0 \leq v \leq t$$

$$= X(v)a(t)\Phi(t, v) + b(t)U(t) + a(t)\int_v^t \Phi(t, w)b(w)U(w)dw$$

$$= a(t)[X(v)\Phi(t, v) + \int_v^t \Phi(t, w)b(w)U(w)dw] + b(t)U(t)$$

$$= a(t)X(t) + b(t)U(t), \quad t \geq 0 \quad \text{(letting } v = t\text{)},$$

which is (9.103). Indeed, (9.107) is a solution to (9.103).

9.20 Letting $t \geq u > v \geq 0$, multiplying (9.107) --- with its mean subtracted --- by $U(u) - m_U$, and taking expected values yields

$$K_{XU}(t, u) = E\{[X(t) - m_X][U(u) - m_U]\}$$
$$= \Phi(t, v)E\{[X(v) - m_X][U(u) - m_U]\}$$
$$+ \int_v^t \Phi(t, w)b(w)E\{[U(w) - m_U][U(u) - m_U]\}dw$$

$$= \Phi(t, v)K_{XU}(v, u) + \int_v^t \Phi(t, w)b(w)K_U(w, u)dw.$$

Using the conditions (9.105) and (9.106) in this equation yields

$$K_{XU}(t, u) = \int_v^t \Phi(t, w)b(w)\delta(w - u)dw, \qquad 0 \le v < u \le t,$$

which is (9.109).

9.21 *a*) The autocovariance of this process is given by (9.114),

$$K_X(t, u) = \begin{cases} \Phi(t, u)K_X(u, u), & t \ge u \ge 0, \\ \Phi(u, t)K_X(t, t), & u \ge t \ge 0, \end{cases}$$

where from (9.108)

$$\Phi(t, u) = \exp\{\int_u^t a \, ds\} = e^{a(t-u)}, \quad t \ge u \ge 0$$

$$\Phi(u, t) = \exp\{\int_t^u a \, ds\} = e^{a(u-t)}, \quad u \ge t \ge 0$$

or, equivalently,

$$\Phi(t, u) = e^{a|t-u|}, \quad t, u \ge 0. \qquad (*)$$

Furthermore, the variance of $X(t)$ must satisfy the differential equation (9.117),

$$\dot{K}_X(t, t) = 2aK_X(t, t) + b^2, \quad t \ge 0$$

with the initial value given by $K_X(0, 0) = 0$. The solution to this differential equation is

$$K_X(t, t) = -\frac{b^2}{2a}[1 - e^{2at}], \quad t \ge 0,$$

and it follows from (9.114) and (*) that the autocovariance is given by

$$K_X(t, u) = \begin{cases} -\frac{b^2}{2a}[e^{a(t-u)} - e^{a(t+u)}], & t \ge u \ge 0 \\ -\frac{b^2}{2a}[e^{a(u-t)} - e^{a(u+t)}], & u \ge t \ge 0, \end{cases}$$

which is equivalent to (9.161).

b) Letting $t, u \to \infty$ in (9.161) yields (since $a < 0$)

$$\lim_{t,u\to\infty} K_X(t,u) = \lim_{t,u\to\infty} k[e^{a|t-u|} - e^{a(t+u)}] = ke^{a|t-u|},$$

which is the desired result (5.22).

c) For $a \to 0$, applying L'Hôpital's rule to (9.161) yields

$$\begin{aligned}\lim_{a\to 0} K_X(t,u) &= \lim_{a\to 0} -b^2 \frac{e^{a|t-u|} - e^{a(t+u)}}{2a} \\ &= \lim_{a\to 0} -b^2 \frac{|t-u| e^{a|t-u|} - (t+u)e^{a(t+u)}}{2} \\ &= -\frac{b^2}{2}[|t-u| - (t+u)] = \frac{b^2}{2}\min\{t,u\}, \qquad t,u \geq 0,\end{aligned}$$

which is the autocovariance for the Wiener process.

9.22 *a*) Using $\mathbf{X}(t)$ as defined in (9.163),

$$\mathbf{X}(t) = \begin{bmatrix} X_1(t) \\ X_2(t) \end{bmatrix} = \begin{bmatrix} Y(t) \\ \dot{Y}(t) \end{bmatrix} \quad \text{or} \quad \begin{aligned} X_1(t) &= Y(t) \\ X_2(t) &= \dot{Y}(t) = \dot{X}_1(t), \end{aligned}$$

(9.162) can be reexpressed as

$$\begin{aligned}\dot{X}_1(t) &= X_2(t), \qquad t \geq 0 \\ \dot{X}_2(t) &= -\frac{1}{LC}X_1(t) - \frac{R}{L}X_2(t) + \frac{1}{LC}U(t), \qquad t \geq 0 \\ Y(t) &= X_1(t), \qquad t \geq 0,\end{aligned}$$

which is of the form of (9.119) with $\mathbf{A}(t)$, $\mathbf{b(t)}$, and $\mathbf{c}(t)$ given by (9.164).

b) Directly applying expectation to (9.162), and interchanging the expectation and differentiation operations, yields

$$\ddot{m}_Y(t) + 6\dot{m}_Y(t) + 8m_Y(t) = 8, \quad t \geq 0.$$

For the given conditions, $K_U(\tau) = \delta(\tau)$ and $K_{\dot{Y}U}(t, v) = K_{YU}(t, v) = 0$, $t < v$, the differential equation for $\mathbf{K_X}(t, t)$ is given by the general formula (9.129):

$$\dot{\mathbf{K}}_{\mathbf{X}}(t,t) = \begin{bmatrix} 0 & 1 \\ -8 & -6 \end{bmatrix} \mathbf{K_X}(t,t) + \mathbf{K_X}(t,t)\begin{bmatrix} 0 & -8 \\ 1 & -6 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 64 \end{bmatrix}$$

It follows from (9.130) that

$$K_Y(t,t) = \mathbf{c}(t)\mathbf{K_X}(t,t)\mathbf{c}^T(t) = K_{X_1}(t,t).$$

Therefore, $K_Y(t, t)$ can be determined once the differential equation for $\mathbf{K_X}(t, t)$ is solved.

9.23 *a*) The correlation matrix for $\mathbf{Y}$ in (9.165) is given by

$$\mathbf{R_Y} = E\{\mathbf{YY}^T\} = \mathbf{\Psi} E\{\mathbf{XX}^T\}\mathbf{\Psi}^T = \mathbf{\Psi R_X \Psi}^T$$

$$= \begin{bmatrix} \mathbf{\Psi}_1^T \\ \mathbf{\Psi}_2^T \\ \cdot \\ \cdot \\ \cdot \\ \mathbf{\Psi}_n^T \end{bmatrix} \mathbf{R_X} \begin{bmatrix} \mathbf{\Psi}_1 & \mathbf{\Psi}_2 & \cdots & \mathbf{\Psi}_n \end{bmatrix} = \begin{bmatrix} \mathbf{\Psi}_1^T\mathbf{R_X}\mathbf{\Psi}_1 & \mathbf{\Psi}_1^T\mathbf{R_X}\mathbf{\Psi}_2 & \cdots & \mathbf{\Psi}_1^T\mathbf{R_X}\mathbf{\Psi}_n \\ \mathbf{\Psi}_2^T\mathbf{R_X}\mathbf{\Psi}_1 & \mathbf{\Psi}_2^T\mathbf{R_X}\mathbf{\Psi}_2 & \cdots & \mathbf{\Psi}_2^T\mathbf{R_X}\mathbf{\Psi}_n \\ \vdots & \vdots & & \vdots \\ \mathbf{\Psi}_n^T\mathbf{R_X}\mathbf{\Psi}_1 & \mathbf{\Psi}_n^T\mathbf{R_X}\mathbf{\Psi}_2 & \cdots & \mathbf{\Psi}_n^T\mathbf{R_X}\mathbf{\Psi}_n \end{bmatrix}$$

Since

$$\mathbf{R_X}\mathbf{\Psi}_i = \lambda_i \mathbf{\Psi}_i \qquad (\text{or } \mathbf{R_X\Psi} = \mathbf{\Lambda\Psi})$$

and

$$\mathbf{\Psi}_i^T\mathbf{\Psi}_j = \begin{cases} 1, & i = j \\ 0, & i \neq j, \end{cases} \qquad (\text{or } \mathbf{\Psi}^T\mathbf{\Psi} = \mathbf{I}),$$

then this correlation matrix becomes

$$\mathbf{R_Y} = \begin{bmatrix} \lambda_1\mathbf{\Psi}_1^T\mathbf{\Psi}_1 & \lambda_2\mathbf{\Psi}_1^T\mathbf{\Psi}_2 & \cdots & \lambda_n\mathbf{\Psi}_1^T\mathbf{\Psi}_n \\ \lambda_1\mathbf{\Psi}_2^T\mathbf{\Psi}_1 & \lambda_2\mathbf{\Psi}_2^T\mathbf{\Psi}_2 & \cdots & \lambda_n\mathbf{\Psi}_2^T\mathbf{\Psi}_n \\ \vdots & \vdots & & \vdots \\ \lambda_1\mathbf{\Psi}_n^T\mathbf{\Psi}_1 & \lambda_2\mathbf{\Psi}_n^T\mathbf{\Psi}_2 & \cdots & \lambda_n\mathbf{\Psi}_n^T\mathbf{\Psi}_n \end{bmatrix}$$

$$= \mathrm{diag}\{\lambda_1, \lambda_2, \ldots, \lambda_n\} \triangleq \mathbf{\Lambda} \qquad (\text{or } \mathbf{\Psi}^T\mathbf{R_X\Psi} = \mathbf{\Lambda\Psi}^T\mathbf{\Psi} = \mathbf{\Lambda}),$$

for which $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. Therefore, $\{Y_i\}$ are mutually orthogonal and their mean square values are in descending order.

b) The correlation matrix for $\mathbf{Z}$ is given by

$$\mathbf{R_Z} = E\{\mathbf{ZZ}^T\} = \mathbf{\Theta} E\{\mathbf{XX}^T\}\mathbf{\Theta}^T = \mathbf{\Theta R_X \Theta}^T = \mathbf{\Psi}^T\mathbf{\Lambda}^{-1/2}\mathbf{\Psi R_X \Psi}^T\mathbf{\Lambda}^{-1/2}\mathbf{\Psi}.$$

It follows from part *a* that $\mathbf{\Psi R_X \Psi}^T = \mathbf{\Lambda}$. Therefore,

$$\mathbf{R_Z} = \mathbf{\Psi}^T\mathbf{\Lambda}^{-1/2}\mathbf{\Lambda\Lambda}^{-1/2}\mathbf{\Psi} = \mathbf{\Psi}^T\mathbf{\Psi} = \mathbf{I}.$$

Hence, $\{Z_1, Z_2, \ldots, Z_n\}$ are mutually orthogonal and their mean square values are all unity.

Chapter 10

Spectral Density

10.1 *a*) Using the change of variables $\tau = t_1 - t_2$ in (10.7) yields

$$E\{\tilde{X}(f_1)\tilde{X}^*(f_2)\} = \iint_{-\infty}^{\infty} e^{-i2\pi(f_1t_1 - f_2t_2)} R_X(t_1 - t_2)dt_1dt_2$$

$$= \iint_{-\infty}^{\infty} e^{-i2\pi(f_2 - f_1)t_2}\, e^{-i2\pi f_1\tau} R_X(\tau)d\tau dt_2$$

$$= \int_{-\infty}^{\infty} R_X(\tau)e^{-i2\pi f_1\tau}d\tau \int_{-\infty}^{\infty} e^{-i2\pi(f_2 - f_1)t_2}dt_2$$

$$= S_X(f_1)\delta(f_2 - f_1),$$

which is the desired result (10.8)

b) Substituting (10.11) and (10.6) into the left-side of (10.142) yields

$$\tilde{X}^{(-1)}(f_1) - \tilde{X}^{(-1)}(f_2) = \int_{-\infty}^{f_1} \tilde{X}(\nu)d\nu - \int_{-\infty}^{f_2} \tilde{X}(\nu)d\nu$$

$$= \int_{f_2}^{f_1} \tilde{X}(\nu)d\nu = \int_{f_2}^{f_1}\int_{-\infty}^{\infty} X(t)e^{-i2\pi\nu t}dtd\nu$$

$$= \int_{-\infty}^{\infty} X(t)[\frac{e^{-i2\pi f_2t} - e^{-i2\pi f_1t}}{i2\pi t}]dt,$$

which is the desired result (10.142)

c) To simplify the notation, we use the change of variables $f_2 = f - \Delta/2$ and $f_1 = f + \Delta/2$. Then, using (10.142), we obtain

$$MS \triangleq E\{|\tilde{X}^{(-1)}(f_1) - \tilde{X}^{(-1)}(f_2)|^2\}$$

$$= \iint_{-\infty}^{\infty} E\{X(t)X(s)\}\frac{\sin(\pi\Delta t)}{\pi t}\frac{\sin(\pi\Delta s)}{\pi s}e^{-i2\pi f(t-s)}dtds,$$

where we have used the identity

$$\frac{e^{i\pi\Delta t} - e^{-i\pi\Delta t}}{2i} = \sin(\pi\Delta t).$$

Thus,

$$MS = \int_{-\infty}^{\infty} \frac{\sin(\pi\Delta t)}{\pi t} \int_{-\infty}^{\infty} [R_X(t - s)e^{-i2\pi f(t-s)}] \frac{\sin(\pi\Delta s)}{\pi s} dsdt,$$

which is of the form

$$MS = \int_{-\infty}^{\infty} a(t)[b(t) \otimes a(t)]^* \, dt.$$

Using Parseval's relation and the convolution theorem, we obtain

$$MS = \int_{-\infty}^{\infty} A(\nu)[B(\nu)A(\nu)]^* \, d\nu,$$

where

$$A(\nu) = \begin{cases} 1, & |\nu| \leq \Delta/2 \\ 0, & otherwise \end{cases} \quad \text{and} \quad B(\nu) = S_X(\nu - f).$$

Therefore,

$$\begin{aligned} MS &= \int_{-\Delta/2}^{\Delta/2} S_X(\nu - f)d\nu = \int_{-f-\Delta/2}^{-f+\Delta/2} S_X(\mu)d\mu \\ &= \int_{f-\Delta/2}^{f+\Delta/2} S_X(\mu)d\mu \qquad (\text{ since } S_X \text{ is even}) \\ &= \int_{f_2}^{f_1} S_X(\mu)d\mu, \end{aligned}$$

which is the desired result (10.143).

d) With the use of (10.12), (10.144) can be expressed (formally) as

$$\tilde{Y}^{(-1)}(f) = \int_{-\infty}^{f} H(\nu)\tilde{X}(\nu)d\nu.$$

Differentiating both sides of this equation yields

$$\tilde{Y}(f) = \frac{d}{df}\tilde{Y}^{(-1)}(f) = \frac{d}{df}\int_{-\infty}^{f} H(\nu)\tilde{X}(\nu)d\nu = H(f)\tilde{X}(f),$$

which is (10.3), as desired.

10.2 *a*) To verify Parseval's relation (10.145), we proceed as follows. Substituting

$$g(t) = \int_{-\infty}^{\infty} G(f)e^{i2\pi ft} df$$

into the left member of (10.145) and interchanging the order of integration yields

$$\int_{-\infty}^{\infty} g(t)h^*(t)dt = \int_{-\infty}^{\infty} G(f)\int_{-\infty}^{\infty} h^*(t)e^{i2\pi ft}\,dtdf$$

$$= \int_{-\infty}^{\infty} G(f)[\int_{-\infty}^{\infty} h(t)e^{-i2\pi ft}\,dt]^*\,df = \int_{-\infty}^{\infty} G(f)H^*(f)df,$$

which is the right member of (10.145).

b) Since from (10.20) we have

$$\int_{-\infty}^{\infty} r_h(\tau)e^{-i2\pi\nu\tau}d\tau = |H(\nu)|^2 \quad \text{and} \quad \int_{-\infty}^{\infty} R_X(\tau)e^{-i2\pi\nu\tau}d\tau = S_X(\nu),$$

then by using Parseval's relation we can express (10.23) as

$$\int_{-\infty}^{\infty} r_h(u)R_X(u)du = \int_{-\infty}^{\infty} |H(\nu)|^2 S_X(\nu)d\nu,$$

which is the desired result, (10.24).

10.3 The autocorrelation for $X(t)$ is given by

$$R_X(\tau) = E\{X(t+\tau)X(t)\} = E\{[Z(t+\tau)-Z(t+\tau-\Delta)][Z(t)-Z(t-\Delta)]\}$$

$$= E\{Z(t+\tau)Z(t)-Z(t+\tau)Z(t-\Delta)-Z(t+\tau-\Delta)Z(t)+Z(t+\tau-\Delta)Z(t-\Delta)\}$$

$$= 2R_Z(\tau) - R_Z(\tau+\Delta) - R_Z(\tau-\Delta) = \alpha^2[2\delta(\tau)-\delta(\tau+\Delta)-\delta(\tau-\Delta)].$$

Fourier transforming $R_X(\tau)$ yields the spectral density for $X(t)$,

$$S_X(f) = \int_{-\infty}^{\infty} R_X(\tau)e^{-i2\pi f\tau}d\tau = 2\alpha^2[1-\cos(2\pi f\Delta)].$$

This result can be understood in terms of linear filtering by interpreting $Z(t)$ and $X(t)$ as the input and output of a linear filter,

$$X(t) = \int_{-\infty}^{\infty} h(t-\tau)Z(\tau)d\tau,$$

with impulse-response function

$$h(t) = \delta(t) - \delta(t-\Delta).$$

The corresponding transfer function has squared magnitude

$$|H(f)|^2 = |1 - e^{-i2\pi f\Delta}|^2 = |i2e^{-i\pi f\Delta}\sin(\pi f\Delta)|^2 = 2[1-\cos(2\pi f\Delta)].$$

Thus,

$$|H(f)|^2 S_Z(f) = 2\alpha^2[1-\cos(2\pi f\,\Delta)].$$

10.4 *a*) Since from (6.40) the Wiener process $W(t)$ can be expressed as

$$W(t) = \int_0^t Z(u)du, \qquad t \geq 0,$$

for which $Z(t)$ is white noise, then $Y(t) = W(t) - W(t-\Delta)$ can be expressed as

$$Y(t) = \int_{t-\Delta}^{t} Z(u)du = \int_0^{\Delta} Z(t-u)du = \int_{-\infty}^{\infty} h(u)Z(t-u)du,$$

where

$$h(t) = \begin{cases} 1, & 0 \leq u < \Delta \\ 0, & \textit{otherwise.} \end{cases}$$

Hence, $Z(t)$ and $Y(t)$ can be viewed as the input and output of a linear filter with impulse-response function $h(t)$. Then, since the input $Z(t)$ is WSS and the filter is stable, the output $Y(t)$ is WSS. The spectral density for $Y(t)$ is given by

$$S_Y(f) = |H(f)|^2 S_Z(f) = \alpha^2\left[\frac{\sin(\pi f\,\Delta)}{\pi f}\right]^2 = 2\alpha^2[1-\cos(2\pi f\,\Delta)]\left[\frac{1}{2\pi f}\right]^2.$$

The spectral density for $Y(t)$ is similar to that of $X(t)$ in exercise 10.3 because $Y(t)$ and $X(t)$ are related by $dY(t)/dt = X(t)$ (cf. part *b*).

b) Fourier transforming $X(t) = dY(t)/dt$ yields

$$\tilde{X}(f) = i2\pi f\tilde{Y}(f)$$

(assuming for the moment that $X(t)$ and $Y(t)$ are Fourier transformable). Therefore, $\tilde{X}(f)$ and $\tilde{Y}(f)$ are related by a linear time-invariant transformation with transfer function $H(f) = i2\pi f$. Consequently, the spectral densities for $X(t)$ and $Y(t)$ are related by

$$S_X(f) = |H(f)|^2 S_Y(f) = (2\pi f)^2 S_Y(f).$$

10.5 *a*) Fourier transforming $R_Y(\tau) = e^{-2\lambda|\tau|}$ yields the following spectral density:

$$S_Y(f) = \int_{-\infty}^{\infty} R_X(\tau)e^{-i2\pi f\tau}d\tau = \int_{-\infty}^{\infty} e^{-2\lambda|\tau|}e^{-i2\pi f\tau}d\tau$$

$$= \int_{-\infty}^{0} e^{2\lambda\tau}e^{-i2\pi f\tau}d\tau + \int_0^{\infty} e^{-2\lambda\tau}e^{-i2\pi f\tau}d\tau$$

$$= \frac{1}{2\lambda - i2\pi f} + \frac{1}{2\lambda + i2\pi f} = \frac{4\lambda}{(2\pi f)^2 + (2\lambda)^2}.$$

b) Use of the result in part *a* with $\lambda = \alpha/2$ yields the Lorenzian spectrum

$$S_X(f) = \sigma^2 \frac{4\lambda}{(2\pi f)^2 + (2\lambda)^2}\bigg|_{\lambda=\alpha/2} = \frac{2\sigma^2\alpha}{(2\pi f)^2 + \alpha^2}$$

for the Ornstein-Uhlenbeck process.

10.6 Substituting (10.34) into (10.38) yields

$$R_X(k) = \int_{-1/2}^{1/2} S_X(f) e^{i2\pi fk}\, df = \sum_{l=-\infty}^{\infty} R_X(l) \int_{-1/2}^{1/2} e^{-i2\pi fl} e^{i2\pi fk}\, df$$

$$= \sum_{l=-\infty}^{\infty} R_X(l) \frac{\sin[\pi(k-l)]}{\pi(k-l)} = R_X(k).$$

Thus, (10.34) and (10.38) are indeed a Fourier transform pair.

10.7 For this derivation, see the solution to exercise 7.4.

10.8 Equation (10.51) can be expressed as

$$W_{1/T}(\nu) = T\left[\frac{\sin(\pi T\nu)}{\pi T\nu}\right]^2 = TH(\nu)H(\nu),$$

where

$$H(\nu) = \frac{\sin(\pi T\nu)}{\pi T\nu}.$$

Inverse Fourier transforming $W_{1/T}(\nu)$ and using the convolution theorem yields

$$\int_{-\infty}^{\infty} W_{1/T}(\nu) e^{i2\pi\nu\tau} d\nu = Th(\tau) \otimes h(\tau),$$

where

$$h(\tau) = \int_{-\infty}^{\infty} H(\nu) e^{i2\pi\nu\tau} d\nu = \begin{cases} 1/T, & |\tau| \le T/2 \\ 0, & \textit{otherwise}, \end{cases}$$

which is easily verified by Fourier transforming $h(\tau)$. Therefore, we have

$$w_T(\tau) = Th(\tau) \otimes h(\tau),$$

which leads to the desired result (10.49) since the convolution of a rectangle with itself is a triangle.

10.9 Since $X(t)$ exhibits mean-square ergodicity of the mean and $S_X(f)$ does not contain an impulse at $f = 0$, then (see (10.141)) we have $m_X = 0$. The mean of the output of the filter with input $X(t)$ is also zero, since

$$m_Y = E\{Y(t)\} = \int_{-\infty}^{\infty} h(u)E\{X(t-u)\}du = 0.$$

The variance of the input is given by

$$Var\{X(t)\} = E\{X^2(t)\} = R_X(0) = \int_{-\infty}^{\infty} S_X(f)df = BS_0.$$

Since the input and output spectral densities are related by

$$S_Y(f) = |H(f)|^2 S_X(f),$$

and the variance at the output is given by

$$Var\{Y(t)\} = E\{Y^2(t)\} = R_Y(0) = \int_{-\infty}^{\infty} S_Y(f)df,$$

then

$$Var\{Y(t)\} = \int_{-\infty}^{\infty} |H(f)|^2 S_X(f)df = \int_{-b}^{b} (1 - |f|/B)S_0 df$$
$$= \begin{cases} bS_0[2 - b/B], & |b| \le B \\ BS_0, & |b| > B. \end{cases}$$

10.10 Let the input to the system be denoted by $X(i)$. We have

$$E\{X(i)\} = 0 \quad \text{and} \quad R_X(i-j) = E\{X(i)X(j)\} = \begin{cases} \sigma^2, & i = j \\ 0, & i \ne j. \end{cases}$$

a) Since the input spectral density is

$$S_X(f) = \sum_{k=-\infty}^{\infty} R_X(k)e^{-i2\pi fk} = \sigma^2$$

and the transfer function is

$$H(f) = \sum_{k=-\infty}^{\infty} h(k)e^{-i2\pi fk} = \sum_{k=0}^{K} e^{-i2\pi fk}$$
$$= \frac{1 - e^{-i2\pi f(K+1)}}{1 - e^{-i2\pi f}} = \frac{\sin[\pi f(K+1)]}{\sin(\pi f)} e^{-i\pi fK},$$

then from (10.33) the output spectral density is given by

$$S_Y(f) = S_X(f)|H(f)|^2 = \sigma^2 \left[\frac{\sin[\pi f (K+1)]}{\sin(\pi f)} \right]^2 .$$

The output variance is given by

$$Var\{Y(i)\} = E\{Y^2(i)\} = \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} h(i-k)h(i-l)E\{X(k)X(l)\}$$

$$= \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} h(i-k)h(i-l)R_X(k-l)$$

$$= \sigma^2 \sum_{k=-\infty}^{\infty} h^2(i-k) = \sigma^2 \sum_{k=0}^{K} h^2(k) = (K+1)\sigma^2.$$

Note: This approach is simpler than that based on the formula

$$Var\{Y(i)\} = \int_{-1/2}^{1/2} S_Y(f)df.$$

b) Since the input and output are related by

$$Y(i) = \sum_{k=-\infty}^{\infty} h(i-k)X(k),$$

then the cross-correlation of the input and output is

$$R_{YX}(i-j) = E\{Y(i)X(j)\} = \sum_{k=-\infty}^{\infty} h(i-k)E\{X(k)X(j)\}$$

$$= \sum_{k=-\infty}^{\infty} h(i-k)R_X(k-j) = \sum_{k=i-K}^{i} R_X(k-j)$$

$$= \sum_{l=i-j-K}^{i-j} \sigma^2\delta(l) = \begin{cases} \sigma^2, & 0 \le i-j \le K \\ 0, & \textit{otherwise.} \end{cases}$$

The cross-spectral density is obtained by Fourier series transformation:

$$S_{YX}(f) = \sum_{k=-\infty}^{\infty} R_{YX}(k)e^{-i2\pi fk} = \sum_{k=0}^{K} \sigma^2 e^{-i2\pi fk} = \sigma^2 \frac{\sin[\pi f (K+1)]}{\sin(\pi f)} e^{-i\pi fK}.$$

10.11 Since the output $Z(i)$ can be expressed as

$$Z(i) = Y(i) - Y(i-k) = \sum_{j=0}^{i} X(j) - \sum_{j=0}^{i-k} X(j)$$

$$= \sum_{j=i-k+1}^{i} X(j) = \sum_{j=0}^{k-1} X(i-j) = \sum_{j=-\infty}^{\infty} X(i-j)h(j),$$

where

$$h(i) = \begin{cases} 1, & 0 \le i \le k-1 \\ 0, & otherwise, \end{cases}$$

then we can identify $h(i)$ as the unit-pulse response of this discrete-time system. Consequently, the transfer function is given by

$$H(f) = \sum_{j=-\infty}^{\infty} h(j) e^{-i2\pi fj} = \frac{\sin(\pi fk)}{\sin(\pi f)} e^{-i\pi f(k-1)}.$$

Therefore, the asymptotic spectral density of $Z(i)$ is given by

$$S_Z(f) = |H(f)|^2 S_X(f) = \sigma_X^2 \left[\frac{\sin(\pi fk)}{\sin(\pi f)}\right]^2.$$

The variance of $Z(i)$ is given by

$$Var\{Z(i)\} = Var\left\{\sum_{j=0}^{k-1} X(i-j)\right\} = k\sigma_X^2.$$

10.12 *a*) Since $R_X(k)$ has even symmetry,

$$R_X(k) = E\{X(i+k)X(i)\} = E\{X(i)X(i+k)\} = R_X(-k),$$

then so too does $S_X(f)$:

$$S_X(f) = \sum_{k=-\infty}^{\infty} R_X(k) e^{-i2\pi fk} = \sum_{k=-\infty}^{\infty} R_X(-k) e^{-i2\pi fk} = \sum_{l=-\infty}^{\infty} R_X(l) e^{-i2\pi(-f)l} = S_X(-f).$$

b) The periodicity of $S_X(f)$ is easily verified as follows:

$$S_X(f+n) = \sum_{k=-\infty}^{\infty} R_X(k) e^{-i2\pi(f+n)k} = \sum_{k=-\infty}^{\infty} R_X(k) e^{-i2\pi fk} e^{-i2\pi nk} = S_X(f)$$

for $n = 0, \pm 1, \pm 2, \ldots$.

c) Analogous to the characterization of the PSD for continuous time in Section 10.2.2, we have for discrete time

$$S_X(f) = \lim_{T\to\infty} \frac{1}{T} E\{|\tilde{X}_T(f)|^2\} \ge 0,$$

where

$$\tilde{X}_T(f) = \frac{1}{2N+1} \sum_{k=-N}^{N} X(k) e^{-i2\pi fk}, \qquad T = 2N+1.$$

Thus, the PSD is nonnegative.

10.13 From (10.19), the spectral densities for $Y_1(t)$ and $Y_2(t)$ are given by

$$S_{Y_1}(f) = |H_1(f)|^2 S_{X_1}(f) \quad \text{and} \quad S_{Y_2}(f) = |H_2(f)|^2 S_{X_2}(f).$$

Since the cross-correlation of $Y_1(t)$ and $Y_2(t)$ is given by

$$R_{Y_1Y_2}(\tau) = E\{Y_1(t+\tau)Y_2(t)\} = \iint_{-\infty}^{\infty} h_1(u)h_2(v)E\{X_1(t+\tau-u)X_2(t-v)\}dudv$$

$$= \iint_{-\infty}^{\infty} h_1(u)h_2(v)R_{X_1X_2}(\tau+v-u)dudv = h_1(\tau)\otimes h_2(-\tau)\otimes R_{X_1X_2}(\tau),$$

then we obtain the following cross-spectral density for $Y_1(t)$ and $Y_2(t)$:

$$S_{Y_1Y_2}(f) = \int_{-\infty}^{\infty} R_{X_1X_2}(\tau)e^{-i2\pi f\tau}d\tau = H_1(f)H_2^*(f)S_{X_1X_2}(f).$$

Thus, the coherence is given by

$$\rho_{Y_1Y_2}(f) = \frac{S_{Y_1Y_2}(f)}{[S_{Y_1}(f)S_{Y_2}(f)]^{1/2}} = \frac{H_1(f)H_2^*(f)}{|H_1(f)||H_2(f)|}\frac{S_{X_1X_2}(f)}{[S_{X_1}(f)S_{X_2}(f)]^{1/2}}$$

$$= \rho_{X_1X_2}(f)\exp\{i[\phi_1(f)-\phi_2(f)]\},$$

where

$$H_1(f) = |H_1(f)|\exp\{i\phi_1(f)\} \quad \text{and} \quad H_2(f) = |H_2(f)|\exp\{i\phi_2(f)\}.$$

Hence, the coherence magnitudes are equal:

$$|\rho_{Y_1Y_2}(f)| \equiv |\rho_{X_1X_2}(f)|.$$

10.14 One way to identify $H(f)$ by using $S_{YX}(f)$ and $S_X(f)$ is to form the ratio

$$\hat{H}(f) = \frac{S_{YX}(f)}{S_X(f)}.$$

It is shown in Chapter 13 that $\hat{H}(f)$ is the best-fitting linear time-invariant model for the unknown system in the sense that $\hat{H}(f)$ minimizes the mean-squared error between the outputs of the unknown system and the model, when both have the same input $X(t)$.

10.15 It follows from the model for $Y_1(t)$ and $Y_2(t)$ that (cf. solution to exercise 10.13)

$$S_{Y_1}(f) = |H_1(f)|^2 S_X(f) + S_{N_1}(f)$$

$$S_{Y_2}(f) = |H_2(f)|^2 S_X(f) + S_{N_2}(f)$$

$$S_{Y_1Y_2}(f) = H_1(f)H_2^*(f)S_X(f).$$

Therefore, the coherence magnitude is given by

$$|\rho_{Y_1Y_2}(f)| \triangleq \frac{|S_{Y_1Y_2}(f)|}{[S_{Y_1}(f)S_{Y_2}(f)]^{1/2}}$$

$$= \frac{|H_1(f)||H_2(f)|S_X(f)}{[|H_1(f)|^2S_X(f) + S_{N_1}(f)]^{1/2}[|H_2(f)|^2S_X(f) + S_{N_2}(f)]^{1/2}}.$$

If, for some cutoff frequency f_0, we have

$$|H_i(f)|^2S_X(f) \gg S_{N_i}(f) \quad \text{for} \quad f \ll f_0, \ i = 1, 2$$

and

$$|H_i(f)|^2S_X(f) \ll S_{N_i}(f) \quad \text{for} \quad f \gg f_0, \ i = 1, 2,$$

then

$$|\rho_{Y_1Y_2}(f)| \simeq \frac{|H_1(f)| \ |H_2(f)|S_X(f)}{[|H_1(f)|^2S_X(f)]^{1/2}[|H_2(f)|^2S_X(f)]^{1/2}} = 1 \quad \text{for} \quad f \ll f_0$$

and

$$|\rho_{Y_1Y_2}(f)| \simeq \frac{|H_1(f)||H_2(f)|S_X(f)}{[S_{N_1}(f)S_{N_2}(f)]^{1/2}} \ll 1 \quad \text{for} \quad f \gg f_0.$$

Also, when the signal $X(t)$ is absent, we have

$$\rho_{Y_1Y_2}(f) = \frac{S_{N_1N_2}(f)}{[S_{N_1}(f)S_{N_2}(f)]^{1/2}} = 0 \quad \text{for all } f.$$

Therefore, the coherence magnitude can be used to detect the presence of the signal $X(t)$.

10.16 *a*) The empirical autocorrelation for the signal in example 1 is given by

$$\hat{R}_X(\tau) = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} X(t+\tau)X(t)dt$$

$$= \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} \frac{1}{2}[\cos(2\pi V\tau) + \cos(4\pi Vt + 2\pi V\tau + 2\Theta)]dt = \frac{1}{2}\cos(2\pi V\tau).$$

Fourier transforming $\hat{R}_X(\tau)$ yields the empirical PSD

$$\hat{S}_X(f) = \int_{-\infty}^{\infty} \hat{R}_X(\tau)e^{-i2\pi f\tau}d\tau = \frac{1}{4}\delta(f - V) + \frac{1}{4}\delta(f + V),$$

which is easily verified by inverse Fourier transforming. The probabilistic autocorrelation for $X(t)$ is given by

$$R_X(\tau) = E\{X(t+\tau)X(t)\} = E\{\frac{1}{2}[\cos(2\pi V\tau) + \cos(4\pi Vt + 2\pi V\tau + 2\Theta)]\}$$

$$= \frac{1}{2}\int_{-\infty}^{\infty}\cos(2\pi v\tau)f_V(v)dv + \frac{1}{2}\int_{-\infty}^{\infty}f_V(v)\int_{-\pi}^{\pi}\frac{1}{2\pi}\cos(4\pi vt + 2\pi v\tau + 2\theta)d\theta\, dv$$

$$= \frac{1}{2}\int_{-\infty}^{\infty}\cos(2\pi v\tau)f_V(v)dv.$$

Fourier transforming $R_X(\tau)$ yields the probabilistic PSD,

$$S_X(f) = \int_{-\infty}^{\infty}R_X(\tau)e^{-i2\pi f\tau}d\tau = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{1}{2}\cos(2\pi v\tau)f_V(v)e^{-i2\pi f\tau}d\tau\, dv$$

$$= \frac{1}{4}\int_{-\infty}^{\infty}[\delta(f-v) + \delta(f+v)]f_V(v)dv = \frac{1}{4}f_V(f) + \frac{1}{4}f_V(-f).$$

Also, since $X(t)$ is WSS, we have the following alternative approach to obtaining $S_X(f)$ (cf. (10.63)):

$$S_X(f) = E\{\hat{S}_X(f)\} = \int_{-\infty}^{\infty}[\frac{1}{4}\delta(f-v)+\frac{1}{4}\delta(f+v)]f_V(v)dv = \frac{1}{4}f_V(f) + \frac{1}{4}f_V(-f).$$

b) Using a similar approach to deriving $R_X(\tau)$ with V replaced by $f_0 + V$ in (10.70) yields

$$R_X(\tau) = \frac{1}{2}\int_{-\infty}^{\infty}\cos(2\pi[f_0+v]\tau)f_V(v)dv.$$

Fourier transforming $R_X(\tau)$ gives the PSD,

$$S_X(f) = \int_{-\infty}^{\infty}R_X(\tau)e^{-i2\pi f\tau}d\tau = \frac{1}{4}f_V(f-f_0) + \frac{1}{4}f_V(-f-f_0).$$

10.17 Using the change of variable $t' = t + \tau/2$ in the left member of the stated identity yields

$$\lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2}R_X(t+\tau,\, t)dt = \lim_{T\to\infty}\frac{1}{T}\int_{-T/2+\tau/2}^{T/2+\tau/2}R_X(t'+\tau/2,\, t'-\tau/2)dt'.$$

Since $T \to \infty$, then $T/2 + \tau/2 \to T/2$ and $-T/2 + \tau/2 \to -T/2$; therefore, we obtain the right member of the identity.

10.18 For a regular nonstationary process $X(t)$, we obtain from (10.1)

$$R_Y(t+\tau,\, t) = E\{Y(t+\tau)Y(t)\} = \iint_{-\infty}^{\infty} h(u)h(v)E\{X(t+\tau-u)X(t-v)\}dudv$$

$$= \iint_{-\infty}^{\infty} h(u)h(v)R_X(t+\tau-u,\ t-v)dudv.$$

It follows that the time-averaged autocorrelations for $X(t)$ and $Y(t)$ are related by

$$<R_Y>(\tau) = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} R_Y(t+\tau,\ t)dt$$

$$= \iint_{-\infty}^{\infty} h(u)h(v) \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} R_X(t+\tau-u,\ t-v)dt\ dudv$$

$$= \iint_{-\infty}^{\infty} h(u)h(v)<R_X>(\tau+v-u)\,dudv = r_h(\tau)\otimes <R_X>(\tau)$$

and, therefore, the time-averaged spectral densities are related by

$$<S_Y>(f) = \int_{-\infty}^{\infty} <R_Y>(\tau)e^{-i2\pi f\tau}d\tau = |H(f)|^2<S_X>(f).$$

10.19 The empirical autocorrelation for the signal in example 2 is given by

$$\hat{R}_X(\tau) = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} X(t+\tau)X(t)dt$$

$$= \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} A(t+\tau)A(t)\frac{1}{2}[\cos(2\pi f_0\tau)+\cos(4\pi f_0 t+2\pi f_0\tau+2\theta)]dt$$

$$= \frac{1}{2}\cos(2\pi f_0\tau)\lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} A(t+\tau)A(t)dt + 0 \quad \text{(using the stated assumption)}$$

$$= \frac{1}{2}\hat{R}_A(\tau)\cos(2\pi f_0\tau),$$

which is (10.79). Fourier transforming $\hat{R}_X(\tau)$ yields the empirical PSD (10.80),

$$\hat{S}_X(f) = \int_{-\infty}^{\infty} \hat{R}_X(\tau)e^{-i2\pi f\tau}d\tau = \frac{1}{4}\hat{S}_A(f-f_0) + \frac{1}{4}\hat{S}_A(f+f_0),$$

which is easily verified by inverse Fourier transforming. The probabilistic autocorrelation for this signal is given by

$$R_X(t+\tau/2,\ t-\tau/2) = E\{X(t+\tau/2)X(t-\tau/2)\}$$

$$= E\{A(t+\tau/2)A(t-\tau/2)\}\frac{1}{2}[\cos(2\pi f_0\tau)+\cos(4\pi f_0 t+2\theta)]$$

$$= \frac{1}{2}R_A(\tau)[\cos(2\pi f_0\tau)+\cos(4\pi f_0 t+2\theta)].$$

Fourier transforming $R_X(t+\tau/2, t-\tau/2)$ yields the instantaneous probabilistic spectral density (10.82),

$$S_X(t,f) = \int_{-\infty}^{\infty} R_X(t+\tau/2, t-\tau/2)e^{-i2\pi f\tau}d\tau$$

$$= \frac{1}{4}S_A(f-f_0) + \frac{1}{4}S_A(f+f_0) + \frac{1}{2}S_A(f)\cos(4\pi f_0 t + 2\theta).$$

10.20 Using the result in exercise 6.12, we obtain the following autocorrelation for the asynchronous random telegraph signal:

$$R_Y(t+\tau, t) = E\{Y(t+\tau)Y(t)\} = E\{X(t+\tau)X(t)\}\cos(2\pi f_0[t+\tau])\cos(2\pi f_0 t)$$

$$= R_X(\tau)\frac{1}{2}[\cos(2\pi f_0\tau) + \cos(4\pi f_0 t + 2\pi f_0\tau)],$$

where

$$R_X(\tau) = e^{-2\lambda|\tau|} \quad \text{for} \quad t+\tau > 0 \text{ and } t > 0.$$

Therefore, the time-averaged autocorrelation for $Y(t)$ is given by

$$\langle R_Y\rangle(\tau) = \lim_{T\to\infty} \frac{1}{T}\int_0^T R_Y(t+\tau, t)dt = \frac{1}{2}R_X(\tau)\cos(2\pi f_0\tau) \quad \text{for} \quad \tau > 0.$$

Since $\langle R_Y\rangle(\cdot)$ must be an even function, this result holds for $\tau < 0$ as well. Fourier transforming $\langle R_Y\rangle(\tau)$ yields the time-averaged PSD for $Y(t)$,

$$\langle S_Y\rangle(f) = \int_{-\infty}^{\infty} \langle R_Y\rangle(\tau)e^{-i2\pi f\tau}d\tau$$

$$= \frac{1}{2}\frac{4\lambda}{(2\pi f)^2 + (2\lambda)^2} \otimes \frac{1}{2}[\delta(f-f_0) + \delta(f+f_0)]$$

$$= \frac{\lambda}{4\pi^2(f-f_0)^2 + 4\lambda^2} + \frac{\lambda}{4\pi^2(f+f_0)^2 + 4\lambda^2}.$$

10.21 The transfer function for the circuit in Figure 10.6 is

$$H(f) = \frac{\tilde{Y}(f)}{\tilde{X}(f)} = \frac{R_2(i2\pi fR_1C + 1)}{R_1 + R_2 + i2\pi fR_1R_2C} = \frac{R_2}{R_1}\frac{i2\pi fR_1C + 1}{(1+R_2/R_1) + i2\pi fR_2C}$$

and the squared magnitude of $H(f)$ can be expressed (using $f_1 = 1/2\pi R_1C \ll f_2 = 1/2\pi R_2C$) as

$$|H(f)|^2 = \left[\frac{R_2}{R_1}\right]^2 \frac{1 + (f/f_1)^2}{(1+R_2/R_1)^2 + (f/f_2)^2} \simeq \left[\frac{R_2}{R_1}\right]^2 [1 + (f/f_1)^2] \quad \text{for} \quad \begin{matrix} f \ll f_2 \\ R_2 \ll R_1 \end{matrix}.$$

Therefore, we have

$$S_Y(f) = |H(f)|^2 S_X(f) \simeq S_0\left[\frac{R_2}{R_1}\right]^2 \quad \text{for} \quad f \ll f_2,$$

which indicates that the high frequencies (well below f_2) are restored to the signal (however, all frequencies passed are strongly attenuated since $R_2 \ll R_1$).

10.22 For nonoverlapping $S_Y(f)$ and $S_X(f)$, we have

$$S_{YX}(f) = \sqrt{S_Y(f)S_X(f)}\rho_{YX}(f) = 0.$$

Since $|\rho_{YX}(f)| \leq 1$, this implies that $S_{YX}(f) \equiv 0$ and, therefore,

$$E\{Y(t+\tau)X(t)\}\} = R_{YX}(\tau) = \int_{-\infty}^{\infty} S_{YX}(f)e^{i2\pi f\tau}df \equiv 0.$$

Therefore, $Y(t+\tau)$ and $X(t)$ are orthogonal for all τ when the spectral densities $S_Y(f)$ and $S_X(f)$ do not overlap.

10.23 This circuit in Figure 10.7 is equivalent to a thermal noise voltage source of value $N(t)$ (in volts) in series with a noise-free resistor of value R and an inductor of value L. It follows that the transfer function from the noise voltage $N(t)$ to the inductor current $X(t)$ is given by

$$\frac{\tilde{X}(f)}{\tilde{N}(f)} = H(f) = \frac{1}{R + i2\pi fL} = \frac{1}{R}\frac{1}{1 + i(f/f_0)}, \quad f_0 = \frac{R}{2\pi L}.$$

Since, from (10.97), $S_N(f) = 2KTR$, then the PSD for the inductor current is given by

$$S_X(f) = |H(f)|^2 S_N(f) = \frac{2KT}{R}\frac{1}{1 + (f/f_0)^2}$$

and the autocorrelation is therefore given by

$$R_X(\tau) = \int_{-\infty}^{\infty} S_X(f)\exp\{i2\pi f\tau\}df = \frac{2KT}{R}\pi f_0 \exp\{-2\pi f_0|\tau|\} = \frac{KT}{L}\exp\{-\frac{R}{L}|\tau|\}.$$

Also, since $Y(t) = L\frac{dX(t)}{dt}$, then the transfer function from the current $X(t)$ to the voltage $Y(t)$ is given by

$$\frac{\tilde{Y}(f)}{\tilde{X}(f)} = H'(f) = i2\pi fL.$$

Hence, the PSD for the inductor voltage is

$$S_Y(f) = |H'(f)|^2 S_X(f) = 2KTR\frac{(f/f_0)^2}{1 + (f/f_0)^2}$$

and the autocorrelation is

$$R_Y(\tau) = \int_{-\infty}^{\infty} S_Y(f) e^{i2\pi f\tau} df = 2KTR\,[\delta(\tau) - \frac{R}{2L}\exp\{-\frac{R}{L}|\tau|\}],$$

which is easily verified by Fourier transforming. From (10.29), we have the cross-spectrum

$$S_{YX}(f) = S_X(f)H'(f) = S_{XY}(-f) = i\,2KTL\frac{f/f_0}{1+(f/f_0)^2}$$

and the cross-correlation

$$R_{YX}(\tau) = R_{XY}(-\tau) = \frac{KTR}{L}[\exp\{\frac{R}{L}\tau\}u(-\tau) - \exp\{-\frac{R}{L}\tau\}u(\tau)]$$

(which is easily verified by Fourier transforming) for the inductor current and voltage.

10.24 Since $S(t)$ and $N(t)$ are orthogonal, then

$$MSE = E\{[S(t) - \hat{S}(t)]^2\} = E\{[S(t) - h(t)\otimes S(t) - h(t)\otimes N(t)]^2\}$$

$$= E\{[S(t) - h(t)\otimes S(t)]^2\} + E\{[h(t)\otimes N(t)]^2\}$$

$$= E\{S^2(t)\} - 2E\{[h(t)\otimes S(t)]S(t)\} + E\{[h(t)\otimes S(t)]^2\} + E\{[h(t)\otimes N(t)]^2\}$$

$$= R_S(0) - 2\int_{-\infty}^{\infty} h(u)R_S(u)du + \iint_{-\infty}^{\infty} h(u)h(v)R_S(v-u)dudv + N_0\iint_{-\infty}^{\infty} h(u)h(v)\delta(v-u)dudv$$

$$= \int_{-\infty}^{\infty} S_S(f)df - \int_{-\infty}^{\infty} H(f)S_S(f)df - \int_{-\infty}^{\infty} H^*(f)S_S(f)df + \int_{-\infty}^{\infty} |H(f)down\,10|^2 S_S(f)df$$

$$+ N_0\int_{-\infty}^{\infty} |H(f)|^2 df$$

$$= \int_{-\infty}^{\infty} |1 - H(f)|^2 S_S(f)df + N_0\int_{-\infty}^{\infty} |H(f)|^2 df,$$

where the next-to-the-last line is obtained by using Parseval's relation and the fact that $H^*(f) = H(-f)$ and $S_S(-f) = S_S(f)$. By expressing $|H(f)|$ as

$$|H(f)| = \left|\frac{1}{1+i(f/f_1)}\right|,$$

which corresponds to (10.101) for $n = 1$ with $b_0 = a_0 = 1$, $b_1 = 0$, and $a_1 = 1/2\pi f_1$, and then expressing $|1 - H(f)||S_S(f)|$ as

$$|1 - H(f)||S_S(f)| = \left|\frac{i(f/f_1)}{1+i(f/f_1)}\right|\left|\frac{\sqrt{S_0}}{1+i(f/f_0)}\right|$$

$$= \left| \frac{i\sqrt{S_0}\,(f/f_1)}{1 + i(f/f_1 + f/f_0) - f^2/f_0 f_1} \right|,$$

which corresponds to (10.101) for $n = 2$ with $b_0 = b_2 = 0$, $b_1 = \sqrt{S_0}/2\pi f_1$, $a_0 = 1$, $a_1 = (1/f_1 + 1/f_0)/2\pi$, and $a_2 = 1/4\pi^2 f_0 f_1$, we can use (10.102b) to obtain

$$MSE = \frac{1}{2}\frac{S_0/(2\pi f_1)^2}{(1/2\pi f_0 + 1/2\pi f_1)/4\pi^2 f_0 f_1} + \frac{1}{2}\frac{N_0}{1/2\pi f_1} = \frac{\pi S_0 f_0^2}{f_0 + f_1} + \pi N_0 f_1.$$

Minimizing this MSE with respect to f_1 yields the desired result:

$$\frac{d}{df_1}MSE = 0 \quad \rightarrow \quad f_1 = f_0[\sqrt{S_0/N_0} - 1],$$

which is an admissible solution for $S_0/N_0 > 1$.

10.25 The estimate $\hat{S}(t)$ can be expressed as

$$\hat{S}(t) = [Y_1(t) + cY_2(t)] \otimes h(t)$$

$$= [S(t) + c\frac{dS(t)}{dt}] \otimes h(t) + [N_1(t) + cN_2(t)] \otimes h(t).$$

Fourier transforming $\hat{S}(t)$ yields

$$\tilde{\hat{S}}(f) = [\tilde{S}(f) + i2\pi f c\tilde{S}(f)]H(f) + [\tilde{N}_1(f) + c\tilde{N}_2(f)]H(f),$$

where

$$H(f) = \frac{1}{1 + i(f/f_1)}.$$

Thus,

$$\tilde{\hat{S}}(f) = \tilde{S}(f) + [\tilde{N}_1(f) + c\tilde{N}_2(f)]H(f) \quad \text{for} \quad c = 1/2\pi f_1.$$

Hence, the estimate $\hat{S}(t)$ is given by

$$\hat{S}(t) = S(t) + [N_1(t) + cN_2(t)] \otimes h(t),$$

which is the undistorted and undelayed signal plus filtered noise. The resultant filtered noise $\hat{N}(t)$ has mean squared value given by

$$E\{\hat{N}^2(t)\} = \int_{-\infty}^{\infty} |H(f)|^2 [N_{01} + c^2 N_{02}]df$$

$$= \int_{-\infty}^{\infty} \frac{1}{1 + (f/f_1)^2}[N_{01} + (\frac{1}{2\pi f_1})^2 N_{02}]df = \pi f_1 N_{01} + \frac{N_{02}}{4\pi f_1}$$

using (10.102a). By equating to zero the derivative of $E\{\hat{N}^2(t)\}$ with respect to f_1, we find that the average power $E\{\hat{N}^2(t)\}$ of the filtered noise is minimized by choosing the cutoff frequency f_1 to satisfy

$$f_1 = \frac{1}{2\pi}\sqrt{N_{02}/N_{01}}.$$

10.26 *a*) The *MSE* is given by

$$MSE = E\{[Y(t)-Z(t)]^2\},$$

where

$$Z(t) \triangleq RC\frac{dX(t)}{dt}.$$

Since $Y(t)$ and $X(t)$ is related by

$$Y(t) + \frac{1}{RC}\int_{-\infty}^{t} Y(u)du = X(t)$$

or, equivalently,

$$\frac{dY(t)}{dt} + \frac{1}{RC}Y(t) = \frac{dX(t)}{dt},$$

then

$$MSE = E\{[Y(t) - Z(t)]^2\} = (RC)^2 E\{[\frac{dY(t)}{dt}]^2\}$$

$$= (RC)^2\int_{-\infty}^{\infty}(i2\pi f)^2 S_Y(f)df = \int_{-\infty}^{\infty}(\frac{f}{f_1})^2 |H(f)|^2 S_X(f)df,$$

where $H(f)$ is the transfer function from $X(t)$ to $Y(t)$,

$$H(f) = \frac{i(f/f_1)}{1+i(f/f_1)}, \qquad f_1 = \frac{1}{2\pi RC}.$$

Thus, the *MSE* is given by

$$MSE = \int_{-\infty}^{\infty}\frac{(f/f_1)^4}{1+(f/f_1)^2}S_X(f)df.$$

But for the $S_X(f)$ given in the exercise, this *MSE* is infinite because $dX(t)/dt$ does not have finite average power. In order to obtain a finite *MSE*, the spectrum of $X(t)$ must decay asymptotically at least as fast as $1/f^4$.

b) We could proceed by analogy with part *a*, but the following is an alternative approach. The *MSE* can be expressed as

$$MSE = E\{[Y(t)-Z(t)]^2\} = E\{Y^2(t)\} - 2E\{Y(t)Z(t)\} + E\{Z^2(t)\},$$

where $Y(t)$ and $Z(t)$ can be expressed as the response of systems $h(t)$ and $w(t)$ to the input signal $X(t)$, respectively,

$$Y(t) = h(t) \otimes X(t) \quad \text{and} \quad Z(t) = w(t) \otimes X(t),$$

in which, according to the circuit diagram,

$$H(f) = \frac{1}{1 + i(f/f_1)} \quad \text{and} \quad W(f) = \frac{1}{i(f/f_1)}.$$

Thus, the MSE can be alternatively written as

$$MSE = \int_{-\infty}^{\infty} [|H(f)|^2 - 2H(f)W^*(f) + |W(f)|^2] S_X(f) df$$

$$= \int_{-\infty}^{\infty} \frac{1}{(f/f_1)^2} \frac{1}{1 + (f/f_1)^2} S_X(f) df.$$

Substituting $S_X(f)$ into this equation yields

$$MSE = S_0 \int_{-\infty}^{\infty} \frac{1}{(f/f_1)^2 [1 + (f/f_1)^2]} \frac{(f/f_0)^2}{1 + (f/f_0)^2} df$$

$$= S_0 \int_{-\infty}^{\infty} \frac{(f_1/f_0)^2}{[1 + (f/f_1)^2][1 + (f/f_0)^2]} df.$$

10.27 *a*) Fourier transforming this differential equation yields

$$(i2\pi f)^n \tilde{Y}(f) + a_{n-1}(i2\pi f)^{n-1}\tilde{Y}(f) + \cdots + a_0 \tilde{Y}(f)$$
$$= b_{n-1}(i2\pi f)^{n-1}\tilde{X}(f) + \cdots + b_0 \tilde{X}(f).$$

The transfer function from $X(t)$ to $Y(t)$ is therefore given by

$$H(f) = \frac{\tilde{Y}(f)}{\tilde{X}(f)} = \frac{b_{n-1}(i2\pi f)^{n-1} + \cdots + b_0}{(i2\pi f)^n + a_{n-1}(i2\pi f)^{n-1} + \cdots + a_0}, \qquad (*)$$

and when $S_X(f) = N_0$ we have

$$S_Y(f) = N_0 |H(f)|^2.$$

The numerator and denominator of $H(f)$ can each be factored into a product of factors of the form $(i2\pi f - z)$, where the root z is either real or complex. For real roots, the numerator and denominator of $|H(f)|^2$ have factors of the form $(2\pi f)^2 + z^2$. For every complex root z, there is another factor with root z^*. In this case, the numerator and denominator of $|H(f)|^2$ have factors of the form $[(2\pi f)^2 + |z|^2] - (4\pi f z_i)^2$, where z_i is the imaginary part of z. Thus, in both cases, the numerator and denominator of $|H(f)|^2$ contain powers of only f^2. Hence $|H(f) down\, 10|^2$ is a rational function of f^2.

b) The discrete-time counterpart of this differential equation is the following difference equation:

$$a_0 Y(i) + a_1 Y(i-1) + \cdots + a_{n-1} Y(i-n+1) + Y(i-n) = b_0 X(i) + \cdots + b_{n-1} X(i-n+1).$$

Fourier series transforming this equation yields

$$a_0 \tilde{Y}(f) + a_1 e^{-i2\pi f} \tilde{Y}(f) + \cdots + a_{n-1} e^{-i2\pi f(n-1)} \tilde{Y}(f) + e^{-i2\pi f n} \tilde{Y}(f) = b_0 \tilde{X}(f) + \cdots + b_{n-1} e^{-i2\pi f(n-1)} \tilde{X}(f),$$

and the transfer function is, therefore, given by

$$H(f) = \frac{\tilde{Y}(f)}{\tilde{X}(f)} = \frac{b_0 + b_1 e^{-i2\pi f} + \cdots + b_{n-1} e^{-i2\pi f(n-1)}}{a_0 + a_1 e^{-i2\pi f} + \cdots + a_{n-1} e^{-i2\pi f(n-1)} + e^{-i2\pi f n}}.$$

Again, when $S_X(f) = N_0$, we have

$$S_Y(f) = |H(f)|^2 N_0,$$

which is the same as (*) with $i2\pi f$ replaced by $e^{i2\pi f}$.

10.28 Since the nth difference of $X(t)$ can be expressed as

$$\begin{aligned}
Y_n(t) &= Y_{n-1}(t) - Y_{n-1}(t-T) \\
Y_{n-1}(t) &= Y_{n-2}(t) - Y_{n-2}(t-T) \\
&\vdots \\
Y_2(t) &= Y_1(t) - Y_1(t-T) \\
Y_1(t) &= X(t) - X(t-T),
\end{aligned}$$

where the transfer function for each of these difference equations is given by

$$H(f) = 1 - e^{-i2\pi f T},$$

then in the frequency domain we have

$$\tilde{Y}_n(f) = H(f)\tilde{Y}_{n-1}(f) = H^2(f)\tilde{Y}_{n-2}(f) = \cdots = H^n(f)\tilde{X}(f).$$

Therefore, the spectral density of the nth difference is given by

$$S_{Y_n}(f) = |H(f)|^{2n} S_X(f),$$

where

$$|H(f)|^{2n} = |1 - e^{-i2\pi f T}|^{2n} = 2^n[1 - \cos(2\pi f T)]^n.$$

10.29 *a*) Since the squared magnitude of the transfer function corresponding to the Hilbert transform is unity,

$$|H(f)|^2 = \left\{ \begin{array}{ll} (-i)(-i)^* = 1, & f > 0 \\ i\, i^* = 1, & f < 0 \end{array} \right\} = 1,$$

then the PSD is unaffected by the Hilbert transform:

$$S_Y(f) = |H(f)|^2 S_X(f) = S_X(f).$$

b) We have, from (10.29), the cross-spectrum

$$S_{XY}(f) = H^*(f) S_X(f);$$

but for the Hilbert transform, $H^*(f) = -H(f)$; therefore,

$$S_{XY}(f) = -H(f) S_X(f),$$

which implies that the cross-correlation $R_{XY}(\tau)$ is the negative of the Hilbert transform of $R_X(\tau)$:

$$R_{XY}(\tau) = -R_X(\tau) \otimes h(\tau),$$

where $h(\tau) = 1/\pi\tau$.

10.30 Fourier transforming the differential equation given in the hint yields

$$m(i2\pi f)^2 \tilde{W}(f) + b(i2\pi f)\tilde{W}(f) + k\tilde{W}(f) + m(i2\pi f)\tilde{V}(f) = 0.$$

From this, we obtain the transfer function relating $V(t)$ and $W(t)$,

$$H(f) = \frac{\tilde{W}(f)}{\tilde{V}(f)} = \frac{-i2\pi fm}{i2\pi fb + k - m(2\pi f)^2}.$$

Hence, the average response power P becomes

$$P = bE\{W^2(t)\} = b\int_{-\infty}^{\infty} S_W(f)df = b\int_{-\infty}^{\infty} |H(f)|^2 S_V(f)df$$

$$= b\int_{-\infty}^{\infty} \left| \frac{i2\pi fm}{i2\pi fb + k - m(2\pi f)^2} \right|^2 \frac{4}{4 + (2\pi f)^2} df.$$

Using (10.102) for $n = 3$ to evaluate P yields (using $b_1 = 2m$, $a_0 = 2k$, $a_1 = 2b + k$, $a_2 = 2m + b$, and $a_3 = m$)

$$P = \frac{2m^2}{4m + 2b + k}.$$

Now, substituting $b^2/4m = k$ into P yields

$$P = \frac{2m^2}{4m + 4\sqrt{mk} + k}.$$

This average power P can be made largest by using the smallest practically feasible spring constant k.

10.31 *a*) Since the noisy resistor can be represented by a thermal noise source $N(t)$ in series with a noise-free resistor, then the transfer function relating $N(t)$ and $Y(t)$ is given by

$$H(f) = \frac{\tilde{Y}(f)}{\tilde{N}(f)} = \frac{1/i2\pi fC}{1/i2\pi fC + i2\pi fL + R} = \frac{1}{1 + i2\pi fRC - (2\pi f)^2 LC}.$$

Therefore, the spectral density for $Y(t)$ when $N(t)$ is white noise with spectral density $N_0 = 1$ is given by

$$S_Y(f) = |H(f)|^2 = \frac{1}{[1-(2\pi f)^2 LC]^2 + (2\pi fRC)^2},$$

which is the desired result.

b) A table of integrals can be used to inverse Fourier transform this PSD; the result is (10.146a).

c) When the losses in the circuit are small, i.e., $R \ll 2\sqrt{L/C}$, then we have

$$\tau_0 f_1 = \frac{2L}{R}\frac{1}{2\pi\sqrt{LC}} = \frac{1}{2\pi}\frac{2\sqrt{L/C}}{R} \gg 1.$$

Therefore, ε in (10.146b) satisfies

$$\varepsilon \ll 1 \quad \text{for} \quad f_0 \simeq f_1$$

and, as a result, (10.146a) reduces to

$$R_Y(\tau) \simeq k\exp\{-|\tau/\tau_0|\}\cos(2\pi f_1\tau).$$

10.32 *a*) From the characterization (10.108) for shot noise, we obtain (using (10.109))

$$E\{Y(t)\} = m_Y = E\{\int_{-\infty}^{\infty} Z(t-u)g(u)du\} = m_Z\int_{-\infty}^{\infty} g(u)du = m_A\lambda G(0)$$

and

$$E\{Y(t+\tau)Y(t)\} = R_Y(\tau) = E\{\iint_{-\infty}^{\infty} g(u)g(v)Z(t+\tau-u)Z(t-v)dudv\}$$

$$= \iint_{-\infty}^{\infty} g(u)g(v)R_Z(\tau+v-u)dudv = R_Z(\tau)\otimes r_g(\tau)$$

$$= [(m_A\lambda)^2 + \lambda(m_A^2+\sigma_A^2)\delta(\tau)]\otimes r_g(\tau)$$

$$= [m_A\lambda]^2 \int_{-\infty}^{\infty} r_g(u)du + \lambda[m_A^2+\sigma_A^2]r_g(\tau)$$

$$= [m_A\lambda G(0)]^2 + \lambda[m_A^2+\sigma_A^2]r_g(\tau).$$

b) For unity pulse amplitudes A_i in the shot noise process, we have $m_A = 1$ and $\sigma_A^2 = 0$. Therefore, part *a* reveals that

$$m_Y = m_A\lambda G(0) = \lambda \int_{-\infty}^{\infty} g(t)dt$$

and

$$\sigma_Y^2 = R_Y(0) - m_Y^2 = [\lambda G(0)]^2 + \lambda r_g(0) - [\lambda G(0)]^2 = \lambda \int_{-\infty}^{\infty} g^2(t)dt.$$

10.33 *a*) From (10.122)-(10.125) we obtain the following mean and autocorrelation for the time-sampled white-noise process:

$$m_Y = E\{Y(i)\} = \int_{-\infty}^{\infty} m_X g(iT-t)dt = 0$$

since $m_X = 0$, and

$$R_Y(k) = E\{Y(i+k)Y(i)\} = \iint_{-\infty}^{\infty} g(iT+kT-u)g(iT-v)R_X(u-v)dudv$$

$$= N_0 \int_{-\infty}^{\infty} g(iT+kT-u)g(iT-u)du$$

$$= N_0 \int_{-\infty}^{\infty} g^2(u)du\, \delta_k, \quad \text{for} \quad \tau_g < T$$

$$= N_0\delta_k \qquad \text{(using (10.125))}.$$

b) From (10.124) and (10.128), we obtain the following autocorrelation for the time-sampled nonwhite-noise process:

$$R_Y(k) = \int_{-\infty}^{\infty} R_X(\tau) \int_{-\infty}^{\infty} g(kT+v-\tau)g(v)dv\ d\tau = \int_{-\infty}^{\infty} R_X(\tau)r_g(\tau-kT)d\tau$$

$$= \delta_k \int_{-\infty}^{\infty} R_X(\tau)[g(\tau)\otimes g(\tau)]d\tau \quad \text{for} \quad \tau_0 < T - \tau_g$$

$$= \delta_k \int_{-\infty}^{\infty} S_X(f)|G(f)|^2 df \qquad \text{(using Parseval's relation)}$$

$$= N_0 \delta_k \qquad \text{(using (10.129)).}$$

10.34 *a*) The autocorrelation for the white noise process $X(t)$ with time-variant attenuation $a(t)$ is given by

$$R_X(t+\tau, t) = E\{X(t+\tau)X(t)\} = a(t+\tau)a(t)E\{Z(t+\tau)Z(t)\}$$
$$= N_0 a(t+\tau)a(t)\delta(\tau) = a^2(t)\delta(\tau),$$

which depends on time t. Therefore, $X(t)$ is nonstationary.

b) The time-averaged probabilistic autocorrelation for $X(t)$ is given by (using the result of part *a*)

$$<R_X>(\tau) = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} R_X(t+\tau, t)dt$$
$$= \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} a^2(t)dt\, \delta(\tau) = <a^2(t)>\delta(\tau)$$

and the corresponding time-averaged spectral density is given by

$$<S_X>(f) = \int_{-\infty}^{\infty} <R_X>(\tau)e^{-i2\pi f\tau}d\tau = <a^2(t)>.$$

10.35 *a*) From Section 10.8, the one-sided bandwidths for the process $X(t)$ with triangular spectrum is given by

$$B_1 = \frac{1}{2}\frac{\int_{-\infty}^{\infty} S_X(f)df}{S_X(0)} = \frac{1}{2}\frac{BS_X(0)}{S_X(0)} = \frac{B}{2},$$

$$B_2 = \frac{1}{2}\left[\frac{\int_{-\infty}^{\infty} f^2 S_X(f)df}{\int_{-\infty}^{\infty} S_X(f)df}\right]^{1/2} = \frac{1}{2}\left[\frac{2S_X(0)B^3/12}{S_X(0)B}\right]^{1/2} = \frac{B}{2\sqrt{6}},$$

and

$$B_3 = \frac{2B}{2} = B.$$

Since the time-averaged PSD for $Y(t) = X(t)\cos(2\pi f_0 t)$ is given by

$$<S_Y>(f) = \frac{1}{4}S_X(f - f_0) + \frac{1}{4}S_X(f + f_0),$$

then the one-sided bandwidths for $Y(t)$ are as follows:

$$B_1 = \frac{\int_0^\infty <S_Y>(f)\,df}{<S_Y>(f_0)} = \frac{\frac{1}{4}S_X(0)B}{\frac{1}{4}S_X(0)} = B,$$

$$B_2 = \left[\frac{\int_0^\infty (f - f_0)^2 <S_Y>(f)\,df}{\int_0^\infty <S_Y>(f)\,df}\right]^{1/2} = \left[\frac{\frac{1}{4}\int_{-B}^{B} f^2 S_X(f)\,df}{\frac{1}{4}S_X(0)B}\right]^{1/2}$$

$$= \left[\frac{\frac{1}{4}S_X(0)B^3/6}{\frac{1}{4}S_X(0)B}\right]^{1/2} = \frac{B}{\sqrt{6}},$$

and

$$B_3 = 2B.$$

b) The mean-squared incremental fluctuation of $X(t)$ is given by

$$E\{[X(t+\tau) - X(t)]^2\} = E\{[X(t) \otimes h(t)]^2\} = R_{X\otimes h}(0) = \int_{-\infty}^{\infty} S_X(f)|H(f)|^2 df,$$

where

$$h(t) = \delta(t+\tau) - \delta(t)$$

and, therefore,

$$H(f) = e^{i2\pi f\tau} - 1.$$

It follows that

$$|H(f)|^2 = [\cos(2\pi f\tau) - 1]^2 + [\sin(2\pi f\tau)]^2 = 2[1 - \cos(2\pi f\tau)] = 4[\sin(\pi f\tau)]^2.$$

Since $S_X(f) = 0$ for $|f| > B_3/2$, then the preceding yields the equation (10.149). It follows from this equation and the inequality $|\sin\phi| \le |\phi|$ that

$$E\{[X(t+\tau) - X(t)]^2\} \le 4\int_{-B_3/2}^{B_3/2} S_X(f)(\pi f\tau)^2 df \le 4\int_{-B_3/2}^{B_3/2} S_X(f)(\pi[B_3/2]\tau)^2 df$$

$$= \int_{-\infty}^{\infty} S_X(f)df\,(\pi B_3\tau)^2 = E\{X^2(t)\}(\pi B_3\tau)^2.$$

The desired result (10.148), which relates the normalized mean-squared incremental fluctuation to the bandwidth, follows immediately.

10.36 White noise is an ergodic process that has a flat spectrum and therefore has no spectral line at $f = 0$. It follows from Section 10.9 that it must have zero mean value.

10.37 *a*) Using Isserlis' formula, we obtain the following autocorrelation for the lag product $Z(t) = X(t)X(t-\tau)$:

$$R_Z(u) = E\{Z(t+u)Z(t)\} = E\{X(t+u)X(t+u-\tau)X(t)X(t-\tau)\}$$
$$= R_X^2(\tau) + R_X^2(u) + R_X(u+\tau)R_X(u-\tau).$$

The spectrum of $Z(t)$ is obtained by Fourier transforming:

$$S_Z(f) = \int_{-\infty}^{\infty} R_Z(u)e^{-i2\pi fu}\,du$$

$$= R_X^2(\tau)\delta(f) + \int_{-\infty}^{\infty} R_X^2(u)e^{-i2\pi fu}\,du + \int_{-\infty}^{\infty} R_X(u+\tau)R_X(u-\tau)e^{-i2\pi fu}\,du$$

$$= R_X^2(\tau)\delta(f) + \int_{-\infty}^{\infty} S_X(\nu)S_X(\nu-f)d\nu + \int_{-\infty}^{\infty} S_X(\nu)e^{i2\pi\nu\tau}S_X(\nu-f)e^{i2\pi(\nu-f)\tau}d\nu$$

(using the convolution theorem and the time-shift property of the Fourier transform). Introducing a change of variables yields the desired result (10.150):

$$S_Z(f) = R_X^2(\tau)\delta(f) + \int_{-\infty}^{\infty} S_X(\nu+f/2)S_X(\nu-f/2)d\nu + \int_{-\infty}^{\infty} S_X(\nu+f/2)S_X(\nu-f/2)e^{i4\pi\nu\tau}d\nu$$

$$= R_X^2(\tau)\delta(f) + \int_{-\infty}^{\infty} S_X(\nu+f/2)S_X(\nu-f/2)[1+\cos(4\pi\nu\tau)]d\nu$$

(since $S_X(\nu+f/2)S_X(\nu-f/2)$ is even and $\sin(4\pi\nu\pi)$ is odd).

b) For $\tau = 0$, (10.150) becomes

$$S_Z(f) = 2\int_{-\infty}^{\infty} S_X(\nu+f/2)S_X(\nu-f/2)d\nu + \delta(f)R_X^2(0)$$

$$= 2S_X(f)\otimes S_X(f) + \delta(f)[\int_{-\infty}^{\infty} S_X(\nu)d\nu]^2.$$

It follows that the width of S_Z is twice that of S_X. That is, squaring the signal doubles its bandwidth.

10.38 *a*) From the decomposition (10.133) and the property (10.134) we obtain the mean

$$m_X = E\{X(t)\} = E\{X_c(t)\} + E\{X_d(t)\} = \sum_n E\{A_n\} e^{i2\pi\alpha_n t}$$

and autocorrelation

$$R_X(t+\tau, t) = E\{X(t+\tau)X(t)\} = E\{X_c(t+\tau)X_c(t)\} + E\{X_d(t+\tau)X_d(t)\}$$
$$= R_{X_c}(\tau) + \sum_n \sum_m E\{A_n A_m\} e^{i2\pi(\alpha_n+\alpha_m)t} e^{i2\pi\alpha_n\tau}.$$

If condition (10.137a) is violated, m_X depends on t and, therefore, $X(t)$ is not WSS. Also, assuming that condition (10.137b) is satisfied, we see (using $\alpha_{-n} = \alpha_n$ and $A_n = A^*_{-n}$) that

$$R_X(t+\tau, t) = R_{X_c}(\tau) + \sum_n E\{|A_n|^2\} e^{i2\pi\alpha_n\tau}; \qquad (*)$$

but if $E\{A_n A_m\} \neq 0$ for some $m \neq -n$, then $R_X(t+\tau, t)$ depends on t and $X(t)$ is not WSS.

b) Let $A_n = |A_n| \exp\{i\Theta_n\}$; then (using the independence of $|A_n|$ and Θ_n)

$$E\{A_n^2\} = E\{|A_n|^2\} E\{\exp[i\Theta_n]\}.$$

If the phase Θ_n is uniformly distributed on $[-\pi, \pi)$, then

$$E\{\exp[i\Theta_n]\} = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i\theta} d\theta = 0,$$

and the condition (10.137b) is satisfied.

c) It follow from (*) and (10.134) that the autocorrelation for the component $X_d(t)$ is given by

$$R_{X_d}(\tau) = \sum_n E\{|A_n|^2\} e^{i2\pi\alpha_n\tau}.$$

Fourier transforming this autocorrelation yields the desired result (10.136) for the PSD.

Chapter 11

Special Topics and Applications

11.1 *a*) The left member of (11.11) can, by interchanging the order of summation and integration and applying the sampling property of the Dirac delta, be expressed as

$$\int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \delta(\tau - nT) e^{-i2\pi f\tau} d\tau = \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} \delta(\tau - nT) e^{-i2\pi f\tau} d\tau = \sum_{n=-\infty}^{\infty} e^{-i2\pi fnT}.$$

Since the right member of (11.11) is periodic with period $1/T$, it can be expressed in a Fourier series,

$$\frac{1}{T} \sum_{m=-\infty}^{\infty} \delta(f - m/T) = \sum_{n=-\infty}^{\infty} c_n e^{i2\pi fnT},$$

where the Fourier coefficients are given by

$$c_n = T \int_{-1/2T}^{1/2T} \frac{1}{T} \sum_{m=-\infty}^{\infty} \delta(f - m/T) e^{-i2\pi nTf} df = \int_{-\infty}^{\infty} \delta(f) e^{-i2\pi nTf} df = 1.$$

This verifies that the left and right members of the identity (11.11) are indeed equal.

b) Applying Parseval's relation to (11.10) and using (11.11) yields

$$S_Z(Tf) = \int_{-\infty}^{\infty} [\int_{-\infty}^{\infty} R_Y^*(\tau) e^{i2\pi f\tau} e^{-i2\pi\nu\tau} d\tau]^* [\int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \delta(\tau - nT) e^{-i2\pi\nu\tau} d\tau] d\nu$$

$$= \int_{-\infty}^{\infty} S_Y(\nu + f) \frac{1}{T} \sum_{m=-\infty}^{\infty} \delta(\nu - m/T) d\nu = \frac{1}{T} \sum_{m=-\infty}^{\infty} S_Y(f + \frac{m}{T})$$

$$= \frac{1}{T} \sum_{m=-\infty}^{\infty} S_Y(f - \frac{m}{T}),$$

which is the desired relation (11.12) between the PSDs of a process and its time-sampled version.

11.2 Fourier transforming (11.21) yields

$$\tilde{Y}(f) = \int_{-\infty}^{\infty} Y(t) e^{-i2\pi ft} dt = \int_{-\infty}^{\infty} \sum_{j=-\infty}^{\infty} X(jT) p(t - jT) e^{-i2\pi ft} dt$$

$$= P(f) \sum_{j=-\infty}^{\infty} X(jT) e^{-i2\pi fjT} = P(f) [\int_{-\infty}^{\infty} X(\tau) \sum_{j=-\infty}^{\infty} \delta(\tau - jT) e^{-i2\pi f\tau} d\tau].$$

By analogy with (11.10) and (11.12), we have

$$\tilde{Y}(f) = \frac{1}{T} P(f) \sum_{k=-\infty}^{\infty} \tilde{X}(f - \frac{k}{T}),$$

which is the desired result (11.22) required to prove the validity of the sampling theorem (11.18)-(11.20).

11.3 *a*) To prove the sampling theorem for mean-square bandlimited WSS processes, we proceed as follows. Using definition (11.263) for $\hat{X}(t)$ yields

$$E\{\hat{X}(t)X(mT)\} = \sum_{n=-\infty}^{\infty} E\{X(nT)X(mT)\}\frac{\sin[(\pi/T)(t-nT)]}{(\pi/T)(t-nT)}$$

$$= \sum_{n=-\infty}^{\infty} R_X([n-m]T)\frac{\sin[(\pi/T)(t-nT)]}{(\pi/T)(t-nT)},$$

and using (11.267) with $b = 0$ results in

$$E\{X(t)X(mT)\} = R_X(t-mT) = \sum_{n=-\infty}^{\infty} R_X([n-m]T)\frac{\sin[(\pi/T)(t-nT)]}{(\pi/T)(t-nT)}.$$

Therefore, we have

$$E\{[X(t)-\hat{X}(t)]X(mT)\} = E\{X(t)X(mT)\} - E\{\hat{X}(t)X(mT)\} = 0$$

for all m, which implies that

$$E\{[X(t)-\hat{X}(t)]\hat{X}(t)\} = \sum_{m=-\infty}^{\infty} E\{[X(t)-\hat{X}(t)]X(mT)\}\frac{\sin[(\pi/T)(t-mT)]}{(\pi/T)(t-mT)} = 0.$$

Also, using (11.263) and (11.267) with $a = \tau = t$ and $b = 0$ yields

$$E\{\hat{X}(t)X(t)\} = \sum_{n=-\infty}^{\infty} E\{X(nT)X(t)\}\frac{\sin[(\pi/T)(t-nT)]}{(\pi/T)(t-nT)}$$

$$= \sum_{n=-\infty}^{\infty} R_X(nT-t)\frac{\sin[(\pi/T)(t-nT)]}{(\pi/T)(t-nT)} = R_X(0).$$

Thus, we also have shown that

$$E\{[X(t)-\hat{X}(t)]X(t)\} = E\{X^2(t)\} - E\{\hat{X}(t)X(t)\} = R_X(0) - R_X(0) = 0.$$

Hence, for a mean-square bandlimited WSS random process, we have shown that

$$E\{[X(t)-\hat{X}(t)]^2\} = E\{[X(t)-\hat{X}(t)]X(t)\} - E\{[X(t)-\hat{X}(t)]\hat{X}(t)\}$$

$$= 0 - 0 = 0,$$

where $\hat{X}(t)$ is the interpolated time-samples of $X(t)$.

b) The MSE in (11.262) can be expressed as

$$MSE = E\{[X(t)-\hat{X}(t)]^2\}$$

$$= E\{X(t)X(t)\} - E\{X(t)\hat{X}(t)\} - E\{\hat{X}(t)X(t)\} + E\{\hat{X}(t)\hat{X}(t)\}$$

$$= R_X(0) - R_{X\hat{X}}(0) - R_{\hat{X}X}(0) + R_{\hat{X}}(0)$$

$$= \int_{-\infty}^{\infty} [S_X(f) - S_{X\hat{X}}(f) - S_{\hat{X}X}(f) + S_{\hat{X}}(f)]df.$$

For $H(f)$ given by (11.270c), we obtain

$$h(t) = \int_{-\infty}^{\infty} H(f)e^{i2\pi ft}df = \frac{\sin[(\pi/T)t]}{(\pi/T)t}.$$

Using $Z(t)$ defined in (11.270b), we obtain

$$\hat{X}(t) = \sum_{n=-\infty}^{\infty} X(nT+\Theta)\frac{\sin[(\pi/T)(t-nT-\Theta)]}{(\pi/T)(t-nT-\Theta)}$$

$$= \sum_{n=-\infty}^{\infty} X(nT+\Theta)\delta(t-nT-\Theta) \otimes h(t) = [X(t)Z(t)] \otimes h(t),$$

which is (11.270a). It then follows that

$$R_{\hat{X}}(\tau) = E\{\hat{X}(t+\tau)\hat{X}(t)\} = E\{X(t+\tau)Z(t+\tau)X(t)Z(t)\} \otimes r_h(\tau) \quad \text{(using (9.45))}$$

$$= [R_Z(\tau)R_X(\tau)] \otimes r_h(\tau)$$

and

$$R_{\hat{X}X}(-\tau) = R_{X\hat{X}}(\tau) = E\{X(t+\tau)\hat{X}(t)\}$$

$$= \int_{-\infty}^{\infty} E\{X(t+\tau)X(t-u)Z(t-u)\}h(u)du = \int_{-\infty}^{\infty} m_Z R_X(\tau+u)h(u)du$$

$$= m_Z R_X(\tau) \otimes h(\tau), \quad \text{(since } h(\tau) \text{ is even)},$$

where, from (11.270b),

$$m_Z = E\{Z(t)\} = \frac{1}{T}\int_{-T/2}^{T/2} \sum_{n=-\infty}^{\infty} \delta(t-nT-\theta)d\theta = \frac{1}{T}\int_{-\infty}^{\infty} \delta(t-\theta)d\theta = \frac{1}{T},$$

and

$$R_Z(\tau) = E\{Z(t+\tau)Z(t)\} = \frac{1}{T}\int_{-T/2}^{T/2} \sum_{n,m=-\infty}^{\infty} \delta(t+\tau-nT-\theta)\delta(t-mT-\theta)d\theta$$

$$= \frac{1}{T}\sum_{n,m=-\infty}^{\infty} \int_{nT-T/2}^{nT+T/2} \delta(t+\tau-\sigma)\delta(t-mT+nT-\sigma)d\sigma \quad \text{(using } \sigma = nT+\theta)$$

$$= \frac{1}{T}\sum_{l=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} \int_{nT-T/2}^{nT+T/2} \delta(t+\tau-\sigma)\delta(t-lT-\sigma)d\sigma \quad \text{(using } l = m-n)$$

$$= \frac{1}{T}\sum_{l=-\infty}^{\infty}\int_{-\infty}^{\infty}\delta(t+\tau-\sigma)\delta(t-lT-\sigma)d\sigma = \frac{1}{T}\sum_{l=-\infty}^{\infty}\delta(\tau-lT).$$

Therefore, we have

$$S_{\hat{X}}(f) = \int_{-\infty}^{\infty}[R_X(\tau)R_Z(\tau)]\otimes r_h(\tau)e^{-i2\pi f\tau}d\tau$$

$$= \int_{-\infty}^{\infty}[R_X(\tau)e^{-i2\pi f\tau}][\frac{1}{T}\sum_{l=-\infty}^{\infty}\delta(\tau-lT)]d\tau\ |H(f)|^2 \quad \text{(using the convolution theorem)}$$

$$= \int_{-\infty}^{\infty}S_X(\nu+f)\frac{1}{T}\sum_{l=-\infty}^{\infty}e^{-i2\pi\nu lT}d\nu|H(f)|^2 \quad \text{(using Parseval's relation)}$$

$$= \int_{-\infty}^{\infty}S_X(\nu+f)\frac{1}{T^2}\sum_{l=-\infty}^{\infty}\delta(\nu-\frac{l}{T})d\nu|H(f)|^2 \quad \text{(using the solution to exc. 11.1)}$$

$$= \frac{1}{T^2}\sum_{l=-\infty}^{\infty}S_X(f+\frac{l}{T})|H(f)|^2 = S_X(f), \quad |f| < 1/2T$$

and

$$S_{\hat{X}X}(-f) = S_{X\hat{X}}(f) = \int_{-\infty}^{\infty}m_Z[R_X(\tau)\otimes h(\tau)]e^{-i2\pi f\tau}d\tau$$

$$= \frac{1}{T}S_X(f)H(f) = S_X(f), \quad |f| < 1/2T.$$

Hence, substituting the preceding two results into (11.268) yields

$$MSE = \int_{-\infty}^{\infty}[S_X(f) - S_X(f) - S_X(f) + S_X(f)]df = 0,$$

which is the desired result: an alternative proof of the sampling theorem for mean-square bandlimited WSS processes.

c) Using an approach similar to that in part *b*, the interpolated nonuniformly spaced samples $\hat{X}(t)$ in (11.274) can be expressed as

$$\hat{X}(t) = [X(t)Z(t)]\otimes h(t),$$

where

$$Z(t) = \sum_{n=1}^{\infty}\delta(t-T_n), \qquad t \geq 0$$

is the sampling process and

$$h(t) = \frac{\sin(2\pi Bt)}{\pi(2B+\lambda)t}$$

is the interpolation pulse, which has Fourier transform

$$H(f) = \begin{cases} 1/(2B+\lambda), & |f| \le B \\ 0, & |f| > B. \end{cases}$$

From (6.43)-(6.44), we obtain

$$m_Z = \lambda, \quad \text{and} \quad R_Z(\tau) = \lambda\delta(\tau) + \lambda^2.$$

Therefore,

$$\begin{aligned} R_{\hat{X}}(\tau) &= [R_Z(\tau)R_X(\tau)] \otimes r_h(\tau) \\ &= [(\lambda\delta(\tau)+\lambda^2)R_X(\tau)] \otimes r_h(\tau) \quad \text{(cf. solution to exc. 6.16)} \end{aligned}$$

$$R_{\hat{X}X}(-\tau) = R_{X\hat{X}}(\tau) = m_Z R_X(\tau) \otimes h(\tau) = \lambda R_X(\tau) \otimes h(\tau).$$

Thus, the *MSE* is given by

$$\begin{aligned} MSE &= \int_{-\infty}^{\infty} [S_X(f) - S_{X\hat{X}}(f) - S^*_{\hat{X}X}(f) + S_{\hat{X}}(f)]df \\ &= \int_{-\infty}^{\infty} \{S_X(f) - \lambda H(f)S_X(f) - \lambda H^*(f)S_X(f) + |H(f)|^2[\lambda R_X(0) + \lambda^2 S_X(f)]\}df \\ &= \int_{-\infty}^{\infty} S_X(f)|1-\lambda H(f)|^2 df + \lambda R_X(0)\int_{-\infty}^{\infty} |H(f)|^2 df \\ &= \left[\frac{2B}{2B+\lambda}\right]^2 R_X(0) + \frac{2\lambda B}{(2B+\lambda)^2}R_X(0) = \frac{1}{1+\lambda/2B}R_X(0). \end{aligned}$$

Hence, the *MSE* decreases as the ratio of the mean sampling rate λ to the bandwidth B increases.

11.4 *a*) The autocorrelation of the pulse-position-modulated signal $X(t)$ in (11.276) is given by

$$\begin{aligned} R_X(t+\tau, t) &= E\{\sum_{j,i=-\infty}^{\infty} p(t+\tau-jT-T_j)p(t-iT-T_i)\} \\ &= \sum_{\substack{j,i=-\infty \\ j\ne i}}^{\infty} E\{p(t+\tau-jT-T_j)\}E\{p(t-iT-T_i)\} \\ &\quad + \sum_{j=-\infty}^{\infty} E\{p(t+\tau-jT-T_j)p(t-jT-T_j)\}. \end{aligned}$$

Since this autocorrelation $R_X(t+\tau, t)$ is periodic in t with period T, then its average value is given by

$$<R_X>(\tau) \triangleq \lim_{Z\to\infty} \frac{1}{Z}\int_{-Z/2}^{Z/2} R_X(t+\tau,\, t)dt = \frac{1}{T}\int_0^T R_X(t+\tau,\, t)dt$$

$$= \sum_{\substack{j,i=-\infty \\ j\neq i}}^{\infty} \frac{1}{T}\int_0^T E\{p(t+\tau-jT-T_j)\}E\{p(t-iT-T_i)\}dt$$

$$+ \sum_{j=-\infty}^{\infty} \frac{1}{T}\int_0^T E\{P(t+\tau-jT-T_j)p(t-jT-T_j)\}dt$$

$$= \sum_{\substack{j,i=-\infty \\ j\neq i}}^{\infty} \frac{1}{T}\int_0^T [\int_{-\infty}^{\infty} p(t+\tau-jT-u)d(u)du][\int_{-\infty}^{\infty} p(t-iT-v)d(v)dv]\, dt$$

$$+ \sum_{j=-\infty}^{\infty} \frac{1}{T}\int_0^T \int_{-\infty}^{\infty} p(t+\tau-jT-u)p(t-jT-u)d(u)du\ dt$$

$$= \sum_{j,i=-\infty}^{\infty} \frac{1}{T}\int_0^T [\int_{-\infty}^{\infty} p(t+\tau-jT-u)d(u)du][\int_{-\infty}^{\infty} p(t-iT-v)d(v)dv]\, dt$$

$$- \sum_{j=-\infty}^{\infty} \frac{1}{T}\int_0^T [\int_{-\infty}^{\infty} p(t+\tau-jT-u)d(u)du][\int_{-\infty}^{\infty} p(t-jT-v)d(v)dv]\, dt$$

$$+ \frac{1}{T}\iint_{-\infty}^{\infty} p(s+\tau-u)p(s-u)d(u)duds \qquad (\text{using } s=t-jT)$$

$$= \frac{1}{T}\sum_{l=-\infty}^{\infty} \int_{-\infty}^{\infty} [\int_{-\infty}^{\infty} p(s+\tau-u)d(u)du][\int_{-\infty}^{\infty} p(s-lT-v)d(v)dv]\, ds$$

$$- \frac{1}{T}\int_{-\infty}^{\infty} [\int_{-\infty}^{\infty} p(s+\tau-u)d(u)du][\int_{-\infty}^{\infty} p(s-v)d(v)dv]\, ds$$

$$+ \frac{1}{T}\int_{-\infty}^{\infty} p(s+\tau)p(s)ds \int_{-\infty}^{\infty} d(u)du \quad (\text{using } l=i-j \text{ and } s=t-jT)$$

$$= \frac{1}{T}\sum_{l=-\infty}^{\infty} \int_{-\infty}^{\infty} w(s+\tau)w(s-lT)ds - \frac{1}{T}\int_{-\infty}^{\infty} w(s+\tau)w(s)ds + \frac{1}{T}r_p(\tau)$$

$$= \frac{1}{T}\sum_{l=-\infty}^{\infty} r_w(\tau+lT) - \frac{1}{T}r_w(\tau) + \frac{1}{T}r_p(\tau),$$

where

$$w(t) = p(t) \otimes d(t).$$

Fourier transforming $<R_X>(\tau)$ yields the time-averaged spectrum

$$<S_X>(f) = \int_{-\infty}^{\infty} <R_X>(\tau) e^{-i2\pi f \tau} d\tau$$

$$= \frac{1}{T} \sum_{l=-\infty}^{\infty} |W(f)|^2 e^{i2\pi f l T} - \frac{1}{T}|W(f)|^2 + \frac{1}{T}|P(f)|^2$$

$$= \left|\frac{P(f)D(f)}{T}\right|^2 \sum_{n=-\infty}^{\infty} \delta(f - \frac{n}{T}) + \frac{1}{T}|P(f)|^2[1 - |D(f)|^2],$$

which is the desired result (11.277).

b) Since $p(t) = e^{-t/T}$ for $t > 0$ and $d(\cdot)$ is a uniform probability density on $[0, a]$, then

$$P(f) = \frac{T}{i2\pi f T + 1} \quad \text{and} \quad D(f) = \frac{\sin(\pi f a)}{\pi f a} e^{-i\pi f a}.$$

Therefore, the time-averaged PSD (11.277) becomes

$$<S_X>(f) = \frac{T}{(2\pi f T)^2 + 1}\left[1 - \left(\frac{\sin(\pi f a)}{\pi f a}\right)^2\right]$$

$$+ \frac{1}{(2\pi f T)^2 + 1}\left[\frac{\sin(\pi f a)}{\pi f a}\right]^2 \sum_{n=-\infty}^{\infty} \delta(f - \frac{n}{T}).$$

11.5 *a*) Inverse Fourier transforming the definition (11.40) yields the relation (11.53),

$$\Gamma(t) = \int_{-\infty}^{\infty} \tilde{\Psi}(f + f_0) e^{i2\pi f t} df$$

$$= \int_{-\infty}^{\infty} \tilde{\Psi}(\nu) e^{i2\pi \nu t} d\nu e^{-i2\pi f_0 t} = \Psi(t) e^{-i2\pi f_0 t} \qquad (\nu = f + f_0),$$

between the complex envelope and analytic signal.

b) Substituting the definition (11.41) into the relation (11.58) yields the relation (11.59),

$$X(t) = \mathrm{Re}\{[U(t) + iV(t)]e^{i2\pi f_0 t}\}$$

$$= \mathrm{Re}\{U(t)\cos(2\pi f_0 t) - V(t)\sin(2\pi f_0 t) + i[U(t)\sin(2\pi f_0 t) + V(t)\cos(2\pi f_0 t)]\}$$

$$= U(t)\cos(2\pi f_0 t) - V(t)\sin(2\pi f_0 t),$$

between the real signal $X(t)$ and its in-phase and quadrature components.

c) From (11.50), we have

$$H(f) = \begin{cases} -i, & f > 0 \\ i, & f < 0. \end{cases}$$

Therefore, since $\tilde{C}(f + f_0) = 0$ for $f > 0$ and $\tilde{C}(f - f_0) = 0$ for $f < 0$, then

$$\int_{-\infty}^{\infty} H\{C(t)\cos(2\pi f_0 t)\} e^{-i 2\pi f t}\, dt = H(f)\frac{1}{2}[\tilde{C}(f + f_0) + \tilde{C}(f - f_0)]$$

$$= \frac{1}{2i}[\tilde{C}(f - f_0) - \tilde{C}(f + f_0)],$$

from which we obtain the Hilbert transform pair

$$H\{C(t)\cos(2\pi f_0 t)\} = \int_{-\infty}^{\infty} \frac{1}{2i}\,[\tilde{C}(f - f_0) - \tilde{C}(f + f_0)] e^{i 2\pi f t}\, df = C(t)\sin(2\pi f_0 t).$$

Similarly, we have

$$\int_{-\infty}^{\infty} H\{S(t)\sin(2\pi f_0 t)\} e^{-i 2\pi f t}\, dt = H(f)\frac{1}{2i}[-\tilde{S}(f + f_0) + \tilde{S}(f - f_0)]$$

$$= -\frac{1}{2}[\tilde{S}(f + f_0) + \tilde{S}(f - f_0)]$$

which results in the Hilbert transform pair

$$H\{S(t)\sin(2\pi f_0 t)\} = \int_{-\infty}^{\infty} -\frac{1}{2}[\tilde{S}(f + f_0) + \tilde{S}(f - f_0)] e^{i 2\pi f t}\, df = -S(t)\cos(2\pi f_0 t).$$

Hence, from (11.57) we obtain

$$\begin{aligned} U(t) &= Z(t)\cos(2\pi f_0 t) + H\{Z(t)\}\sin(2\pi f_0 t) \\ &= C(t)\cos^2(2\pi f_0 t) - S(t)\sin(2\pi f_0 t)\cos(2\pi f_0 t) \\ &\quad + C(t)\sin^2(2\pi f_0 t) + S(t)\cos(2\pi f_0 t)\sin(2\pi f_0 t) = C(t) \end{aligned}$$

and

$$\begin{aligned} V(t) &= H\{Z(t)\}\cos(2\pi f_0 t) - Z(t)\sin(2\pi f_0 t) \\ &= C(t)\sin(2\pi f_0 t)\cos(2\pi f_0 t) + S(t)\cos^2(2\pi f_0 t) \\ &\quad - C(t)\cos(2\pi f_0 t)\sin(2\pi f_0 t) + S(t)\sin^2(2\pi f_0 t) = S(t), \end{aligned}$$

which reveals that $C(t)$ and $S(t)$ in (11.62) are the in-phase and quadrature components (assuming $C(t)$ and $S(t)$ are bandlimited to $(-f_0, f_0)$).

11.6 *a*) Since

$$Y(t) = h(t) \otimes X(t),$$

then $Y(t)$ and $X(t)$ are jointly WSS and

$$E\{X(t)\} = m_X = 0, \qquad E\{Y(t)\} = m_Y = 0,$$

$$E\{X(t+\tau)X(t)\} = R_X(\tau),$$

$$E\{Y(t+\tau)Y(t)\} = R_X(\tau) \otimes r_h(\tau) = R_X(\tau) \quad (\text{since } r_h(\tau) = \delta(\tau)),$$

$$E\{Y(t+\tau)X(t)\} = R_{YX}(\tau) = R_X(\tau) \otimes h(\tau),$$

$$E\{X(t+\tau)Y(t)\} = R_{XY}(\tau) = R_{YX}(-\tau) = -R_{YX}(\tau).$$

Therefore, it follows from (11.57) that

$$E\{U(t)\} = m_U = m_X \cos(2\pi f_0 t) + m_Y \sin(2\pi f_0 t) = 0,$$

$$\begin{aligned} E\{U(t+\tau)U(t)\} &= R_U(\tau) \\ &= R_X(\tau)\cos(2\pi f_0[t+\tau])\cos(2\pi f_0 t) + R_{XY}(\tau)\cos(2\pi f_0[t+\tau])\sin(2\pi f_0 t) \\ &\quad + R_{YX}(\tau)\sin(2\pi f_0[t+\tau])\cos(2\pi f_0 t) + R_Y(\tau)\sin(2\pi f_0[t+\tau])\sin(2\pi f_0 t) \\ &= R_X(\tau)\cos(2\pi f_0\tau) + R_{YX}(\tau)\sin(2\pi f_0\tau) \quad (\text{using trigonometry identities}), \end{aligned}$$

$$E\{V(t)\} = m_V = m_Y \cos(2\pi f_0 t) - m_X \sin(2\pi f_0 t) = 0,$$

$$\begin{aligned} E\{V(t+\tau)V(t)\} &= R_V(\tau) \\ &= R_Y(\tau)\cos(2\pi f_0[t+\tau])\cos(2\pi f_0 t) - R_{YX}(\tau)\cos(2\pi f_0[t+\tau])\sin(2\pi f_0 t) \\ &\quad - R_{XY}(\tau)\sin(2\pi f_0[t+\tau])\cos(2\pi f_0 t) + R_X(\tau)\sin(2\pi f_0[t+\tau])\sin(2\pi f_0 t) \\ &= R_X(\tau)\cos(2\pi f_0\tau) + R_{YX}(\tau)\sin(2\pi f_0\tau) \quad (\text{using trigonometry identities}), \end{aligned}$$

and

$$\begin{aligned} E\{U(t+\tau)V(t)\} &= R_{UV}(\tau) = E\{V(t)U(t-\tau)\} = R_{VU}(-\tau) \\ &= R_{XY}(\tau)\cos(2\pi f_0[t+\tau])\cos(2\pi f_0 t) - R_X(\tau)\cos(2\pi f_0[t+\tau])\sin(2\pi f_0 t) \\ &\quad + R_Y(\tau)\sin(2\pi f_0[t+\tau])\cos(2\pi f_0 t) - R_{YX}(\tau)\sin(2\pi f_0[t+\tau])\sin(2\pi f_0 t) \\ &= R_X(\tau)\sin(2\pi f_0\tau) - R_{YX}(\tau)\cos(2\pi f_0\tau) \quad (\text{using trigonometry identities}). \end{aligned}$$

Thus, the in-phase and quadrature components, $U(t)$ and $V(t)$, of $X(t)$ are jointly WSS if $X(t)$ is WSS.

b) The autocorrelation of $Z(t)$ in (11.62) is given by

$$\begin{aligned} R_Z(t+\tau, t) &= E\{Z(t+\tau)Z(t)\} \\ &= R_C(\tau)\cos(2\pi f_0[t+\tau])\cos(2\pi f_0 t) - R_{CS}(\tau)\cos(2\pi f_0[t+\tau])\sin(2\pi f_0 t) \\ &\quad - R_{SC}(\tau)\sin(2\pi f_0[t+\tau])\cos(2\pi f_0 t) + R_S(\tau)\sin(2\pi f_0[t+\tau])\sin(2\pi f_0 t) \end{aligned}$$

$$= \frac{1}{2}[R_C(\tau)+R_S(\tau)]\cos(2\pi f_0\tau) + \frac{1}{2}[R_C(\tau)-R_S(\tau)]\cos(2\pi f_0[2t+\tau])$$

$$- \frac{1}{2}[R_{SC}(\tau)-R_{CS}(\tau)]\sin(2\pi f_0\tau) - \frac{1}{2}[R_{CS}(\tau)+R_{SC}(\tau)]\sin(2\pi f_0[2t+\tau]).$$

Therefore, $Z(t)$ is WSS if and only if

$$R_{CS}(\tau) = -R_{SC}(\tau) \quad \text{and} \quad R_C(\tau) = R_S(\tau),$$

in which case the resultant autocorrelation is given by (11.63).

c) Since

$$R_{CS}(-\tau) = R_{SC}(\tau) = -R_{CS}(\tau) \quad \text{and} \quad R_{SC}(-\tau) = R_{CS}(\tau) = -R_{SC}(\tau),$$

then

$$S_{CS}(f) = -S_{CS}^*(f) \quad \text{and} \quad S_{SC}(f) = -S_{SC}^*(f),$$

which implies that these cross spectra are purely imaginary. Therefore, $S_Z(f)$ in (11.64) is real, as it must be.

d) As an alternative to the approach to verifying (11.279), for the PSDs of the in-phase and quadrature components, which is outlined in the exercise statement (and which exploits the fact that both equations in (11.57) are of the same form as that in (11.62)), we use the results of part *a*. From part *a*, we have

$$S_U(f) = S_V(f) = \int_{-\infty}^{\infty} [R_X(\tau)\cos(2\pi f_0\tau)+R_{YX}(\tau)\sin(2\pi f_0\tau)]e^{-i2\pi f\tau}d\tau$$

$$= \frac{1}{2}[S_X(f-f_0)+S_X(f+f_0)] + \frac{1}{2i}[S_{YX}(f-f_0)-S_{YX}(f+f_0)].$$

Since from (11.61) we have

$$S_{YX}(f-f_0) = \begin{cases} -iS_X(f-f_0), & f > f_0 \\ iS_X(f-f_0), & f < f_0 \end{cases}$$

$$= -iS_X(f-f_0)u(f-f_0) + iS_X(f-f_0)u(-f+f_0)$$

and

$$S_{YX}(f+f_0) = \begin{cases} -iS_X(f+f_0), & f > -f_0 \\ iS_X(f+f_0), & f < -f_0 \end{cases}$$

$$= -iS_X(f+f_0)u(f+f_0) + iS_X(f+f_0)u(-f-f_0),$$

then the preceding formulas for $S_U(f)$ and $S_V(f)$ reduce to (11.279). It is a simple matter to show that (11.279) yields (11.65) under condition (11.66).

11.7 *a*) See the solution to exercise 11.6*a* for the formula for $R_{UV}(\tau)$.

b) Fourier transforming the result for $R_{UV}(\tau)$ obtained in exercise 11.6*a* yields

$$S_{UV}(f) = -S_{VU}(f) = \int_{-\infty}^{\infty} [R_X(\tau)\sin(2\pi f_0\tau) - R_{YX}(\tau)\cos(2\pi f_0\tau)]e^{-i2\pi f\tau}d\tau$$

$$= \frac{1}{2i}[S_X(f-f_0) - S_X(f+f_0)] - \frac{1}{2}[S_{YX}(f-f_0) + S_{YX}(f+f_0)].$$

But

$$S_{YX}(f) = H(f)S_X(f) = \begin{cases} -iS_X(f), & f > 0 \\ iS_X(f), & f < 0 \end{cases}$$

and

$$S_X(f+f_0) = 0 \quad \text{for} \quad f > f_0 \qquad \text{and} \qquad S_X(f-f_0) = 0 \quad \text{for} \quad f < -f_0.$$

Thus, the cross-spectrum for the in-phase and quadrature components is given by

$$S_{UV}(f) = -S_{VU}(f) = \begin{cases} i[S_X(f+f_0) - S_X(f-f_0)], & |f| < f_0 \\ 0, & |f| > f_0. \end{cases}$$

11.8 *a*) Using trigonometry identities to expand the polar form (11.70) yields the rectangular form

$$X(t) = A(t)\cos[\Phi(t)]\cos(2\pi f_0 t) - A(t)\sin[\Phi(t)]\sin(2\pi f_0 t),$$

which reduces to

$$X(t) = U(t)\cos(2\pi f_0 t) - V(t)\sin(2\pi f_0 t)$$

since, from (11.71)-(11.72),

$$\sin[\Phi(t)] = \frac{V(t)}{A(t)} \qquad \text{and} \qquad \cos[\Phi(t)] = \frac{U(t)}{A(t)}.$$

b) Using formula (11.73) for an FM signal $X(t)$, we obtain the autocorrelation

$$R_X(t+\tau/2, t-\tau/2) = E\{X(t+\tau/2)X(t-\tau/2)\}$$

$$= a^2E\{\cos[2\pi f_0(t+\tau/2) + \Phi(t+\tau/2)]\cos[2\pi f_0(t-\tau/2) + \Phi(t-\tau/2)]\}$$

$$= \frac{a^2}{2}E\{\cos[4\pi f_0 t + \Phi(t+\tau/2) + \Phi(t-\tau/2)] + \cos[2\pi f_0\tau + \Phi(t+\tau/2) - \Phi(t-\tau/2)]\}$$

$$= \frac{a^2}{2}E\{\cos(4\pi f_0 t)\cos[\Phi(t+\tau/2) + \Phi(t-\tau/2)] - \sin(4\pi f_0 t)\sin[\Phi(t+\tau/2) + \Phi(t-\tau/2)]$$

$$+\cos(2\pi f_0\tau)\cos[\Phi(t+\tau/2)-\Phi(t-\tau/2)]-\sin(2\pi f_0\tau)\sin[\Phi(t+\tau/2)-\Phi(t-\tau/2)]\}$$

$$= \frac{a^2}{2}[a(t,\tau)\cos(2\pi f_0\tau)-b(t,\tau)\sin(2\pi f_0\tau)+c(t,\tau)\cos(4\pi f_0 t)$$

$$-d(t,\tau)\sin(4\pi f_0 t)],$$

where $a(t,\tau)$, $b(t,\tau)$, $c(t,\tau)$, and $d(t,\tau)$ are defined by (11.77).

c) Substituting (11.78) into (11.76) and time-averaging the result yields the following time-averaged autocorrelation for an FM signal:

$$\langle R_X\rangle(\tau) = \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2} R_X(t+\tau/2,\ t-\tau/2)dt$$

$$= \frac{a^2}{2}[a(\tau)\cos(2\pi f_0\tau)-b(\tau)\sin(2\pi f_0\tau)]$$

$$+\frac{a^2}{2}[c(\tau)\langle\cos(4\pi f_0 t)\rangle-d(\tau)\langle\sin(4\pi f_0 t)\rangle]$$

$$= \frac{a^2}{2}[a(\tau)\cos(2\pi f_0\tau)-b(\tau)\sin(2\pi f_0\tau)].$$

11.9 To evaluate formula (11.80) for a zero-mean Gaussian process $\Phi(t)$, we use (2.38) with $\omega = [1,\ \ -1]^T$ and $\mathbf{X} = [\Phi(t+\tau/2),\ \ \Phi(t-\tau/2)]^T$. Then from (2.38),

$$\mathbf{m_X} = E\{\mathbf{X}\} = 0 \quad \text{and} \quad \mathbf{K_X} = E\{\mathbf{XX}^T\} = \begin{bmatrix} R_\Phi(0) & R_\Phi(\tau) \\ R_\Phi(\tau) & R_\Phi(0) \end{bmatrix}.$$

Therefore,

$$a(\tau) = \exp\{-\frac{1}{2}\omega^T\mathbf{K_X}\omega\} = \exp\left\{-\frac{1}{2}[1\quad -1]\begin{bmatrix} R_\Phi(0) & R_\Phi(\tau) \\ R_\Phi(\tau) & R_\Phi(0) \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right\}$$

$$= \exp\{-[R_\Phi(0)-R_\Phi(\tau)]\}.$$

11.10 Since from (11.77) and (11.78) we have

$$a(t,0) = a(0) = 1 \quad \text{and} \quad b(t,0) = b(0) = 0,$$

then from (11.79) the time-averaged power of the FM signal is given by

$$<R_X>(0) = \frac{a^2}{2}[a(0)\cos(0) - b(0)\sin(0)] = \frac{a^2}{2},$$

regardless of the modulating signal $\Phi(t)$.

11.11 From (11.75) and (7.23), we obtain the following formulas for the derivatives of the autocorrelation for the phase:

$$R_{\Phi}^{(2)}(0) = -R_{\Psi}(0) = -\int_{-\infty}^{\infty} S_{\Psi}(f)\,df$$

and

$$R_{\Phi}^{(4)}(0) = R_{\Psi}^{(2)}(0) = \int_{-\infty}^{\infty} (2\pi f)^2 S_{\Psi}(f)\,df.$$

Now, using the definitions of Δf, B_{Ψ}, and β in (11.86)-(11.88) yields alternative expressions

$$R_{\Phi}^{(2)}(0) = -(2\pi)^2 \Delta f^2$$

and

$$R_{\Phi}^{(4)}(0) = (2\pi)^2 B_{\Psi}^2 \int_{-\infty}^{\infty} S_{\Psi}(f)\,df = (2\pi)^4 B_{\Psi}^2 \Delta f^2 = (2\pi)^4 \frac{\Delta f^4}{\beta^2}.$$

Substituting these expressions into (11.98) and rearranging the equation yields

$$\begin{aligned} R_{\Phi}(0) - R_{\Phi}(\tau) &= -\frac{1}{2} R_{\Phi}^{(2)}(0)\tau^2 - \frac{1}{24} R_{\Phi}^{(4)}(0)\tau^4 + \cdots \\ &= \frac{1}{2}(2\pi)^2(\Delta f\,\tau)^2 - \frac{(2\pi)^4}{24}\frac{(\Delta f\,\tau)^4}{\beta^2} + \cdots \end{aligned}$$

for the first few terms of the infinite series expansion of $R_{\Phi}(\tau)$.

11.12 Since $\Phi(t)$ is a zero-mean stationary Gaussian process, which means that the probability densities of linear combinations such as $\Theta = \Phi(t+\tau/2) \pm \Phi(t-\tau/2)$ are even functions, and since $\sin[\Phi(t+\tau/2) \pm \Phi(t-\tau/2)]$ is an odd function of Θ, then $b(\tau)$ and $d(\tau)$ in (11.77) are zero. Also, using the result in exercise 11.9 with $\boldsymbol{\omega} = [1, \ \ 1]^T$ yields

$$c(\tau) = \exp\{-\frac{1}{2}\boldsymbol{\omega}^T \mathbf{K_X} \boldsymbol{\omega}\} = \exp\left\{-\frac{1}{2}[1 \quad 1]\begin{bmatrix} R_{\Phi}(0) & R_{\Phi}(\tau) \\ R_{\Phi}(\tau) & R_{\Phi}(0) \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$$

$$= \exp\{-[R_{\Phi}(0) + R_{\Phi}(\tau)]\}.$$

Therefore, substituting $a(\tau)$, $b(\tau)$, $c(\tau)$, and $d(\tau)$ into (11.76) yields

$$R_X(t+\tau/2,\ t-\tau/2) = \frac{a^2}{2}[\exp\{-[R_\Phi(0)-R_\Phi(\tau)]\}\cos(2\pi f_0\tau)$$

$$+ \exp\{-[R_\Phi(0)+R_\Phi(\tau)]\}\cos(4\pi f_0 t)],$$

which is the desired formula (11.102) for the autocorrelation of an FM signal with a zero-mean Gaussian WSS phase.

11.13 From (11.124), we obtain

$$\beta = \left[\frac{1}{3}\frac{SNR_{out}}{SNR_{in}}\right]^{1/3} = \frac{10}{3^{1/3}} \simeq 6.93$$

for the modulation index. Then, since $B_\Psi = 15 \times 10^3$ Hz, the rms frequency deviation is given by (using (11.85))

$$\Delta f = \beta B_\Psi = \frac{10}{3^{1/3}} \times 15 \times 10^3 \simeq 1.04 \times 10^5 \text{ Hz.}$$

Therefore, the rms value of $\Psi(t)$ is (using (11.86) or (11.111))

$$\sqrt{R_\Psi(0)} = \Big[\int_{-\infty}^{\infty} S_\Psi(f)df\Big]^{1/2} = 2\pi\Delta f \simeq 6.53 \times 10^5,$$

the amplitude of the FM signal is (using (11.109))

$$a = (4N_0\Delta f \times SNR_{in})^{1/2} = \frac{(1.5)^{1/2}}{3^{1/6}} \times 2 \times 10^{-1} \simeq 0.204,$$

and the rms bandwidth of the FM signal is (using (11.95))

$$B_X \simeq \Delta f = 1.04 \times 10^5 \text{ Hz.}$$

11.14 *a*) The transfer function for the pre-emphasis circuit in Figure 11.12(a) is given by

$$H_p(f) = \frac{i2\pi fR_1R_2C + R_2}{i2\pi fR_1R_2C + R_1 + R_2},$$

which corresponds to an all-pass filter with some low-frequency attenuation. The transfer function for the de-emphasis circuit in Figure 11.12(b) is given by

$$H_d(f) = \frac{1}{i2\pi fR_1C + 1},$$

which corresponds to a low-pass filter.

b) The noise power at the output of the demodulator without de-emphasis is given by (11.122). The noise power at the output of the demodulator with de-emphasis is given by (11.126) with (11.119) and (11.127) substituted in:

$$N_{out,d} = \int_{-B_\Psi}^{B_\Psi} |H_d(f)|^2 S_Y(f)\,df = 2N_0\left(\frac{2\pi}{a}\right)^2 \int_{-B_\Psi}^{B_\Psi} \frac{f^2}{1+(f/B_d)^2}\,df$$

$$= 4N_0\left(\frac{2\pi}{a}\right)^2 B_d^2[B_\Psi - B_d \tan^{-1}(B_\Psi/B_d)].$$

Therefore, we have the ratio

$$\frac{N_{out}}{N_{out,d}} = \frac{(B_\Psi/B_d)^3}{3[B_\Psi/B_d - \tan^{-1}(B_\Psi/B_d)]},$$

which is the desired result (11.128).

11.15 *a)* Since $S_X(\nu)$ is approximately constant throughout the passband of $H(\nu)$, then

$$[S_X(\nu)|H(\nu)|^2] \otimes [S_X(\nu)|H(\nu)|^2] \simeq \frac{1}{(2\Delta f)^2} S_X^2(f)2[\Delta f - |\nu|]$$

$$= \frac{1}{2\Delta f} S_X^2(f)[1 - \frac{|\nu|}{\Delta f}], \quad |\nu| \le B.$$

Substituting this result into (11.143), with $G(\nu)$ given in (11.144), yields the result

$$\sigma_z^2 \simeq \frac{1}{\Delta f} S_X^2(f) \int_{-1/2\Delta t}^{1/2\Delta t} (1 - \frac{|\nu|}{\Delta f})\,d\nu$$

for the variance of the estimate of the PSD.

b) Using m_Z from (11.137) and σ_Z^2 from (11.145) yields the result

$$\frac{\sigma_Z^2}{m_Z^2} \simeq \frac{1}{\Delta f} \int_{-1/2\Delta t}^{1/2\Delta t} (1 - \frac{|\nu|}{\Delta f})\,d\nu = \frac{1}{\Delta t \Delta f}(1 - \frac{1}{4}\frac{1}{\Delta t \Delta f})$$

for the coefficient of variation of the estimate of the PSD. If we desire $\sigma_Z^2/m_Z^2 < 1/100$, then for $\Delta f = 1$ KHz we require $\Delta t > 9.97 \times 10^{-2}$ sec $\simeq 0.1$ sec. The formula (11.143) used to obtain this result is valid only for Gaussian random processes. However, it is a good approximation for many other types of processes when $\Delta t \Delta f \gg 1$.

For Δt given, in order to obtain $\sigma_Z^2/m_Z^2 < 1/100$, Δf must satisfy

$$\frac{1}{\Delta t \Delta f}(1 - \frac{1}{4}\frac{1}{\Delta t \Delta f}) < \frac{1}{100},$$

which is approximated by

$$\frac{1}{\Delta t \Delta f} < \frac{1}{100},$$

and therefore reduces to $\Delta f > 100/\Delta t$, which limits the spectral resolution.

11.16 *a*) Since the input and output of the circuit in Figure 11.9 are related by

$$V_0(t) = \frac{R_L}{R + R_L} V(t) = V(t) \otimes h(t),$$

where

$$h(t) = \frac{R_L}{R + R_L} \delta(t),$$

then the (resistance-normalized) PSDs of the input and output are related by

$$S_{V_0}(f) = |H(f)|^2 S_V(f) = \left[\frac{R_L}{R + R_L}\right]^2 S_V(f).$$

Therefore, we have the relation

$$S_{V_0}(f)df = \left[\frac{R_L}{R + R_L}\right]^2 S_V(f)df$$

for the increments of the (resistance-normalized) power.

b) Differentiating the right side of (11.160) with respect to R_L and equating the derivative to zero yields

$$\left[\frac{1}{(R + R_L)^2} - \frac{2R_L}{(R + R_L)^3}\right] S_V(f)df = 0 \quad \Longrightarrow \quad R_L = R.$$

Thus, when $R = R_L$, the incremental power $dP_{V_0}(f)$ is maximized.

11.17 *a*) Let T_1 and $V_1(t)$ be the noise temperature and voltage for resistor R_1, and let T_2 and $V_2(t)$ be the noise temperature and voltage for R_2; then the output voltage $V_0(t)$ of the two parallel-connected resistors is given by (using Thevenin equivalent circuits, cf. Section 10.7.2)

$$V_0(t) = \frac{R_2}{R_1 + R_2} V_1(t) + \frac{R_1}{R_1 + R_2} V_2(t).$$

Since we can assume that $V_1(t)$ and $V_2(t)$ are independent thermal noise sources, the PSD at the output is given by

$$S_{V_0}(f) = \left(\frac{R_2}{R_1 + R_2}\right)^2 S_{V_1}(f) + \left(\frac{R_1}{R_1 + R_2}\right)^2 S_{V_2}(f).$$

From (10.88) we obtain

$$2KT_e R_e = \left(\frac{R_2}{R_1 + R_2}\right)^2 2KT_1 R_1 + \left(\frac{R_1}{R_1 + R_2}\right)^2 2KT_2 R_2,$$

where T_e is the effective noise temperature for the parallel connection and R_e is the corresponding effective resistance and is given by

$$R_e = \frac{R_1 R_2}{R_1 + R_2}.$$

Therefore, T_e is given by

$$T_e = \frac{R_2}{R_1 + R_2} T_1 + \frac{R_1}{R_1 + R_2} T_2.$$

b) From (11.167), we obtain the requirement

$$G_1 = \frac{T_2 G_2 + T_3}{(T_e - T_1) G_2} = \frac{400 \times 6 + 800}{(250 - 200) \times 6} = \frac{32}{3} = 10.67$$

on the gain of the first stage.

c) Since $T_L = 280$ K and $L = 1.5$, then from (11.170b) we have

$$T_2 = T_L (L - 1) = 140 \text{ K}.$$

For $G_1 = 1$, $G_2 = 1/L$, $T_1 = T_{ea} = 150$, and $T_3 = T_{er} = 700$, we obtain from (11.167) the effective input noise temperature,

$$T_e = T_1 + \frac{T_2}{G_1} + \frac{T_3}{G_1 G_2} = 150 + \frac{140}{1} + \frac{700}{1/1.5} = 1340 \text{ K}.$$

For $G_r(f_0) = 10^{12}$ and $B_r = 1$ MHz, we obtain from (11.174) the available output noise power for the overall system,

$$\begin{aligned} P_0 &= K[\frac{T_1}{L} + T_L(1 - \frac{1}{L}) + T_3] G_r(f_0) B_r \\ &= 1.38 \times 10^{-23} [\frac{150}{1.5} + 280(1 - \frac{1}{1.5}) + 700] \times 10^{12} \times 1 \times 10^6 \\ &= 1.23 \times 10^{-2} \quad \text{watts}. \end{aligned}$$

11.18 *a*) Applying Parseval's relation to (11.179) yields the frequency-domain formula (11.182):

$$\begin{aligned} E\{Y(t_0) | s(t) \text{ present}\} &= \int_{-\infty}^{\infty} h(t) s(t_0 - t) dt = \int_{-\infty}^{\infty} H(f) [S^*(f) e^{-i2\pi f t_0}]^* df \\ &= \int_{-\infty}^{\infty} H(f) S(f) e^{i2\pi f t_0} df. \end{aligned}$$

Applying the convolution theorem and Parseval's relation to (11.181) yields the frequency-domain formula (11.183):

$$Var\{Y(t_0)|s(t) \text{ present}\} = \iint_{-\infty}^{\infty} R_N(\tau)h(t+\tau)h(t)dt\, d\tau = \int_{-\infty}^{\infty} R_N(\tau)r_h(\tau)d\tau$$

$$= \int_{-\infty}^{\infty} S_N(f)|H(f)|^2 df.$$

b) Substituting (11.188) into (11.182) and (11.183) yields the explicit formulas

$$E\{Y(t_0)|s(t) \text{ present}\} = c\int_{-\infty}^{\infty} \frac{|S(f)|^2}{S_N(f)} df$$

$$Var\{Y(t_0)|s(t) \text{ present}\} = c^2\int_{-\infty}^{\infty} \frac{|S(f)|^2}{S_N(f)} df.$$

Therefore, from (11.178) we have the following explicit formula for the maximum SNR:

$$SNR_{\max} = \max\left\{\frac{[E\{Y(t_0)|s(t) \text{ present}\}]^2}{Var\{Y(t_0)|s(t) \text{ present}\}}\right\} = \int_{-\infty}^{\infty} \frac{|S(f)|^2}{S_N(f)} df.$$

11.19 For $S_N(f) = N_0$, the SNR formula (11.189) reduces to

$$SNR_{\max} = \int_{-\infty}^{\infty} \frac{|S(f)|^2}{S_N(f)} df = \frac{1}{N_0}\int_{-\infty}^{\infty} s^2(t)dt = \frac{E}{N_0}$$

by using Parserval's relation. Using (11.188) with $c = 1$, the impulse-response function reduces to

$$h(t) = \int_{-\infty}^{\infty} H(f)e^{i2\pi ft} df$$

$$= \frac{1}{N_0}\Big[\int_{-\infty}^{\infty} S(f)e^{i2\pi f(t_0-t)} df\Big]^* = \frac{1}{N_0}s^*(t_0-t) = \frac{1}{N_0}s(t_0-t).$$

11.20 For discrete-time signals, (11.178) becomes

$$SNR = \frac{[E\{Y(i_0)|s(t) \text{ present}\}]^2}{Var\{Y(i_0)|s(i) \text{ present}\}},$$

for which

$$E\{Y(i_0)|s(i) \text{ present}\} = \sum_{i=-\infty}^{\infty} h(i)s(i_0-i) = \int_{-1/2}^{1/2} H(f)S(f)e^{i2\pi fi_0} df$$

and

$$Var\{Y(i_0)|s(t)\text{ present}\} = E\{[N(i_0)\otimes h(i_0)]^2\} = \sum_{j,i=-\infty}^{\infty} R_N(j)h(i+j)h(i)$$

$$= \int_{-1/2}^{1/2} |H(f)|^2 S_N(f)df,$$

where

$$H(f) = \sum_{j=-\infty}^{\infty} h(j)e^{-i2\pi fj}$$

$$S(f) = \sum_{j=-\infty}^{\infty} s(j)e^{-i2\pi fj}$$

$$S_N(f) = \sum_{j=-\infty}^{\infty} R_N(j)e^{-i2\pi fj}.$$

Now, let us define

$$G(f) \triangleq H(f)\sqrt{S_N(f)} \quad \text{and} \quad K(f) \triangleq \frac{S^*(f)}{\sqrt{S_N(f)}}e^{-i2\pi fi_0};$$

then the SNR can be expressed as

$$SNR = \frac{[\int_{-1/2}^{1/2} G(f)K^*(f)df]}{\int_{-1/2}^{1/2} |G(f)|^2 df},$$

which can be maximized by applying the Cauchy-Schwarz inequality. The necessary and sufficient condition for a maximum is

$$G(f) = cK(f), \quad c = constant,$$

which is equivalent to

$$H(f) = \frac{cS^*(f)}{S_N(f)}e^{-i2\pi fi_0}.$$

Thus, the unit-pulse response of this optimum discrete-time linear time-invariant filter is given by

$$h(j) = \int_{-1/2}^{1/2} H(f)e^{i2\pi fj}df.$$

11.21 The matched filter is given by

$$h(t) = s(t_0 - t),$$

and the output of the filter is given by

$$y(t) = \int_{-\infty}^{\infty} h(t-u)s(u)du = \int_{-\infty}^{\infty} s(t_0 - t + u)s(u)du$$

$$= \begin{cases} Ta^2[2 - 3\dfrac{|t-t_0|}{T}], & |t-t_0| \le T \\ Ta^2[\dfrac{|t-t_0|}{T} - 2], & T < |t-t_0| \le 2T \\ 0, & \textit{otherwise.} \end{cases}$$

This output clearly reaches its maximum at $t = t_0$, as desired.

11.22 *a*) Substituting the signal estimate (11.195) into the orthogonality condition (11.196) yields

$$E\{[\hat{S}(t) - S(t)]X(u)\} = E\{\hat{S}(t)X(u)\} - E\{S(t)X(u)\}$$

$$= \int_{-\infty}^{\infty} h(v)E\{X(t-v)X(u)\}dv - R_{SX}(t-u)$$

$$= \int_{-\infty}^{\infty} h(v)R_X(t-u-v)dv - R_{SX}(t-u) = 0,$$

which is equivalent to

$$\int_{-\infty}^{\infty} h(v)R_X(\tau - v)dv = R_{SX}(\tau), \qquad \tau = t - u.$$

This is (11.197). Fourier transforming (11.197) yields

$$S_X(f)H(f) = S_{SX}(f),$$

which has the solution

$$H(f) = \frac{S_{SX}(f)}{S_X(f)}.$$

This is the noncausal Wiener filter (11.198).

b) Since $S(t)$ and $N(t)$ are uncorrelated, then from the model (11.193) we obtain the correlation functions

$$R_{SX}(\tau) = E\{S(t+\tau)X(t)\} = \int_{-\infty}^{\infty} g(v)E\{S(t+\tau)S(t-v)\}dv + E\{S(t+\tau)N(t)\}$$

$$= \int_{-\infty}^{\infty} g(v) R_S(\tau+v) dv + 0 = R_S(\tau) \otimes g(-\tau)$$

and

$$R_X(\tau) = E\{X(t+\tau)X(t)\}$$

$$= \iint_{-\infty}^{\infty} g(v)g(u)E\{S(t+\tau-v)S(t-u)\}dudv + E\{N(t+\tau)N(t)\}$$

$$= \iint_{-\infty}^{\infty} g(v)g(u)R_S(\tau+u-v)dudv + R_N(\tau) = R_S(\tau)\otimes r_g(\tau) + R_N(\tau).$$

Fourier transforming these two correlation functions yields formulas (11.199) and (11.200), respectively, for the spectral densities. Finally, substituting (11.199)-(11.200) into (11.198) yields the desired result (11.201) for the transfer function of the Wiener filter.

11.23 Substituting the signal estimate (11.195) and using the orthogonality condition (11.196) in the *MSE* (11.194) yields

$$MSE_{\min} = E\{[S(t)-\hat{S}(t)]^2\} = E\{[S(t)-\hat{S}(t)]S(t)\} - E\{[S(t)-\hat{S}(t)]\hat{S}(t)\}$$

$$= E\{S^2(t)\} - E\{S(t)\hat{S}(t)\} + \int_{-\infty}^{\infty} E\{[\hat{S}(t)-S(t)]X(u)\}h(t-u)du$$

$$= E\{S^2(t)\} - E\{S(t)\hat{S}(t)\},$$

which is the desired result (11.204). Substituting (11.195) into (11.204) yields the more explicit formula (11.205),

$$MSE_{\min} = R_S(0) - \int_{-\infty}^{\infty} h(t-u)E\{S(t)X(u)\}du = R_S(0) - \int_{-\infty}^{\infty} h(t-u)R_{SX}(t-u)du$$

$$= R_S(0) - \int_{-\infty}^{\infty} h(\tau)R_{SX}(\tau)d\tau, \quad \tau = t-u$$

which, by using Parseval's relation, can be reexpressed in the frequency domain as (11.206). Further substitution of (11.198) results in (11.207a),

$$MSE_{\min} = \int_{-\infty}^{\infty} [S_S(f) - \frac{|S_{SX}(f)|^2}{S_X(f)}]df = \int_{-\infty}^{\infty} S_S(f)[1-|\rho(f)|^2]df,$$

and, with (11.199) and (11.200) substituted in, we obtain (11.207b) and (11.207c):

$$MSE_{\min} = \int_{-\infty}^{\infty} [S_S(f) - \frac{|G(f)|^2 S_S^2(f)}{|G(f)|^2 S_S(f) + S_N(f)}]df$$

$$= \int_{-\infty}^{\infty} \frac{S_S(f)S_N(f)}{|G(f)|^2 S_S(f) + S_N(f)} df = \int_{-\infty}^{\infty} \frac{S_S(f)}{1 + r(f)} df,$$

for which $r(f)$ is defined in (11.203).

11.24 *a*) From (11.198), we obtain the Wiener filter for the model given in this exercise:

$$H(f) = \frac{S_{SX}(f)}{S_X(f)} = \frac{S_S(f)}{S_S(f) + S_N(f)} = \begin{cases} \dfrac{1 - |f|/B}{1 - |f|/B + N_0/S_0}, & |f| \leq B \\ 0, & |f| > B. \end{cases}$$

For $|f| \leq B$, the Wiener filter attenuates the signal and noise, especially for $|f| \to B$ and N_0/S_0 not much smaller than unity, whereas the ideal low-pass filter passes the signal and noise; both filters reject the noise for $|f| > B$.

b) From the general formula (11.206), we obtain

$$MSE_{\min} = \int_{-\infty}^{\infty} \frac{S_S(f)S_N(f)}{S_S(f) + S_N(f)} df = \int_{-B}^{B} \frac{S_0(1 - |f|/B)N_0}{S_0(1 - |f|/B) + N_0} df$$

$$= N_0 \int_{-B}^{B} [1 - \frac{N_0}{S_0} \frac{B}{(N_0/S_0 + 1)B - |f|}] df = 2N_0 B \, [1 - \frac{N_0}{S_0} \ln(1 + \frac{S_0}{N_0})],$$

which results in the normalized $MSE_{\min}$

$$\frac{MSE_{\min}}{E\{S^2(t)\}} = \frac{MSE_{\min}}{S_0 B} = \frac{2N_0}{S_0}[1 - \frac{N_0}{S_0} \ln(1 + \frac{S_0}{N_0})].$$

c) When an arbitrary filter $h(t)$ is used, the MSE is given by

$$MSE = E\{[S(t) - \hat{S}(t)]^2\} = E\{S^2(t)\} + E\{\hat{S}^2(t)\} - 2E\{S(t)\hat{S}(t)\}$$

$$= R_S(0) + R_{\hat{S}}(0) - 2\int_{-\infty}^{\infty} R_S(u)h(u)du$$

$$= \int_{-\infty}^{\infty} S_S(f)df + \int_{-\infty}^{\infty} |H(f)|^2 [S_S(f) + S_N(f)]df - 2\int_{-\infty}^{\infty} S_S^*(f)H(f)df. \qquad (*)$$

Substituting the ideal low-pass filter transfer function $H(f) = H_*(f)$ given in part *b* into (*) yields

$$MSE = 2\int_{-B}^{B} S_S(f)df + \int_{-B}^{B} S_N(f)df - 2\int_{-B}^{B} S_S(f)df = \int_{-B}^{B} S_N(f)df = 2BN_0,$$

which results in the normalized MSE

$$\frac{MSE}{E\{S^2(t)\}} = \frac{2N_0}{S_0}.$$

This MSE is larger than $MSE_{\min}$, especially when N_0/S_0 is not much smaller than unity. On the other hand, this ideal low-pass filter does not produce any signal distortion, whereas the Wiener filter distorts the signal by attenuating high-frequency components.

d) Since the estimation error decomposes as

$$S(t) - \hat{S}(t) = [1 - h(t)] \otimes S(t) - h(t) \otimes N(t),$$

then the MSE decomposes as

$$MSE = MSE_S + MSE_N,$$

where the components are given by

$$MSE_S = \int_{-\infty}^{\infty} |D(f)|^2 S_S(f) df$$

$$MSE_N = \int_{-\infty}^{\infty} |A(f)|^2 S_N(f) df,$$

and $D(f)$ and $A(f)$ are effective signal-distortion and noise-attenuation transfer functions:

$$D(f) = 1 - H(f) \quad \text{and} \quad A(f) = -H(f).$$

For the ideal low-pass filter, we have

$$D(f) = 0, \qquad |f| \le B$$

$$A(f) = \begin{cases} -1, & |f| \le B \\ 0, & |f| > B, \end{cases}$$

whereas for the Wiener filter, we have

$$D(f) = \frac{N_0/S_0}{1 + N_0/S_0 - |f|/B}$$

$$A(f) = -\frac{1 - |f|/B}{1 - |f|/B + N_0/S_0} = \frac{-1}{1 + \dfrac{N_0/S_0}{1 - |f|/B}}.$$

Thus, the Wiener filter yields smaller MSE by attenuating the noise more than the ideal low-pass filter does (especially for $|f| \to B$), but at the cost of introducing signal distortion.

11.25 *a*) Using the hint we obtain

$$MSE = E\{[S(t) - \hat{S}(t)]^2\} = E\{[S'(t) - \hat{S}'(t)]^2\},$$

where

$$S'(t) = N_1(t),\ \ \hat{S}'(t) = X'(t) \otimes h(t),\ \text{ and }\ X'(t) = N_1(t) - N_2(t).$$

Therefore, the Wiener filter that minimizes the MSE is given by (11.198) with S and X replaced by S' and X',

$$H(f) = \frac{S_{S'X'}(f)}{S_{X'}(f)} = \frac{S_{N_1}(f)}{S_{N_1}(f) + S_{N_2}(f)}.$$

b) For the noise PSDs given in this case, we have

$$H(f) = \begin{cases} 1, & f \text{ for which } S_{N_1}(f) \neq 0 \text{ and } S_{N_2}(f) = 0 \\ 0, & f \text{ for which } S_{N_2} \neq 0 \text{ and } S_{N_1}(f) = 0. \end{cases}$$

Therefore, the estimate $\hat{S}(t)$ becomes

$$\hat{S}(t) = S(t) + N_1(t) - h(t) \otimes [N_1(t) - N_2(t)] = S(t) + N_1(t) - N_1(t) = S(t)$$

and the signal $S(t)$ is perfectly recovered.

11.26 By using (2.45), the MSE can be expressed as

$$MSE = E\{E\{[\hat{X}(t-\Theta) - X(t-\Theta)]^2|\Theta\}\} = \frac{1}{T}\int_0^T E\{[\hat{X}(t-\theta) - X(t-\theta)]^2\}d\theta.$$

To minimize the MSE, the error $\hat{X}(t-\Theta) - X(t-\Theta)$ must be orthogonal to the measurement $Y(u-\Theta)$ (cf. (11.196)),

$$E\{[\hat{X}(t-\Theta) - X(t-\Theta)]Y(u-\Theta)\} = 0$$

or, equivalently,

$$\frac{1}{T}\int_0^T \int_{-\infty}^{\infty} h(v)E\{Y(t-\theta-v)Y(u-\theta)\}dv d\theta = \frac{1}{T}\int_0^T E\{X(t-\theta)Y(u-\theta)\}d\theta. \qquad (*)$$

Since $X(t)$ is WSS, then

$$\frac{1}{T}\int_0^T E\{Y(t-\theta-v)Y(u-\theta)\}d\theta$$

$$= \frac{1}{T}\int_0^T \sum_{j,i=-\infty}^{\infty} R_X([j-i]T)p(t-\theta-v-jT)p(u-\theta-iT)d\theta$$

$$= \frac{1}{T}\int_0^T \sum_{j,k=-\infty}^{\infty} R_X(kT)p(t-\theta-v-jT)p(u-\theta-kT-jT)d\theta \quad (\text{using } i-j=k)$$

$$= \frac{1}{T}\sum_{k=-\infty}^{\infty} R_X(kT)\int_{-\infty}^{\infty} p(t-\phi-v)p(u-\phi-kT)d\phi \quad (\text{using } \phi = \theta + jT)$$

$$= \frac{1}{T} \sum_{k=-\infty}^{\infty} R_X(kT) \int_{-\infty}^{\infty} (t-u-v+kT+\sigma)p(\sigma)d\sigma, \quad \text{(using } \sigma = u - \phi - kT\text{)}$$

$$= \frac{1}{T} \sum_{k=-\infty}^{\infty} R_X(kT) r_p(t-u-v+kT)$$

and

$$\frac{1}{T}\int_0^T E\{X(t-\theta)Y(u-\theta)\}d\theta = \frac{1}{T}\int_0^T \sum_{j=-\infty}^{\infty} R_X(t-\theta-jT)p(u-\theta-jT)d\theta$$

$$= \frac{1}{T}\int_{-\infty}^{\infty} R_X(t-\phi)p(u-\phi)d\phi \quad \text{(using } \phi = \theta + jT\text{)}$$

$$= \frac{1}{T}\int_{-\infty}^{\infty} R_X(t-u+\sigma)p(\sigma)d\sigma \quad \text{(using } \sigma = u - \phi\text{)}$$

$$= \frac{1}{T} R_X(t-u) \otimes p(-[t-u]).$$

Substituting these two results with $\tau = t - u$ into the orthogonality condition (*) yields

$$\sum_{k=-\infty}^{\infty} R_X(kT) r_p(\tau - kT) \otimes h(\tau - kT) = R_X(\tau) \otimes p(-\tau). \tag{**}$$

Fourier transforming (**) yields

$$H(f)|P(f)|^2 \bar{S}_X(f) = S_X(f)P^*(f),$$

for which

$$\bar{S}(f) = \sum_{k=-\infty}^{\infty} R_X(kT)e^{-i2\pi f kT} = \frac{1}{T}\sum_{k=-\infty}^{\infty} S_X(f-k/T).$$

Thus, the filter that minimizes the MSE is given by

$$H(f) = \frac{S_X(f)}{\frac{1}{T}P(f)\sum_{k=-\infty}^{\infty} S_X(f-k/T)},$$

for all f for which $P(f) \neq 0$, and $H(f)$ is arbitrary for all f for which $P(f) = 0$. The minimum MSE is then given by (cf. (11.204))

$$MSE_{\min} = E\{[X(t-\Theta) - \hat{X}(t-\Theta)]X(t-\Theta)\}$$

$$= E\{E\{[X(t-\Theta) - \hat{X}(t-\Theta)]X(t-\Theta)|\Theta\}\}$$

$$= R_X(0) - \frac{1}{T}\int_0^T E\{[Y(t-\theta) \otimes h(t-\theta)]X(t-\theta)\}d\theta$$

$$= R_X(0) - \frac{1}{T}\int_0^T \int_{-\infty}^{\infty} h(t-v)E\{Y(v-\theta)X(t-\theta)\}dv d\theta$$

$$= R_X(0) - \frac{1}{T}\int_{-\infty}^{\infty} h(t-v)[R_X(t-v) \otimes p(-[t-v])]dv$$

$$= R_X(0) - \frac{1}{T}\int_{-\infty}^{\infty} h(u)[R_X(u) \otimes p(-u)]du$$

$$= \int_{-\infty}^{\infty} [S_X(f) - \frac{1}{T}H(f)P(f)S_X(f)]df$$

$$= \int_{-\infty}^{\infty} S_X(f)[1 - \frac{1}{T}\frac{S_X(f)}{\bar{S}_X(f)}]df = \int_{-\infty}^{\infty} S_X(f)[1 - \frac{S_X(f)}{\sum_{k=-\infty}^{\infty} S_X(f-k/T)}]df$$

(where Parseval's relation and the convolution theorem were used in the next-to-the-last line). This result is independent of $p(t)$ as long as $P(f) \neq 0$ for all f for which $S_X(f) \neq 0$.

If $S_X(f) = 0$ for $|f| > 1/2T$, then $MSE_{\min}$ is zero

$$MSE_{\min} = \int_{-1/2}^{1/2} S_X(f)[1 - \frac{S_X(f)}{S_X(f)}]df = 0$$

and

$$H(f) = \frac{T}{P(f)}.$$

11.27 Using the model (11.208) and the detection statistic Y specified by (11.209), we obtain

$$E\{Y|S(t) \text{ present}\} = E\{\iint_{-T/2}^{T/2} k(u,v)X(u)X(v)dudv\}$$

$$= \iint_{-T/2}^{T/2} k(u,v)E\{[S(u)+N(u)][S(v)+N(v)]\}dudv$$

$$= \iint_{-T/2}^{T/2} k(u,v)[R_S(u,v)+N_0\delta(u-v)]dudv$$

$$E\{Y|S(t) \text{ absent}\} = E\{\iint_{-T/2}^{T/2} k(u,v)X(u)X(v)dudv\}$$

$$= \iint_{-T/2}^{T/2} k(u,v)E\{N(u)N(v)\}dudv$$

$$= \iint_{-T/2}^{T/2} k(u, v) N_0 \delta(u - v) \, du dv.$$

and

$$E\{Y^2 | S(t) \text{ absent}\} = E\{ \iint_{-T/2}^{T/2} k(u, v) X(u) X(v) du dv \iint_{-T/2}^{T/2} k(w, z) X(w) X(z) dw dz \}$$

$$= \iiiint_{-T/2}^{T/2} k(u, v) k(w, z) E\{N(u) N(v) N(w) N(z)\} du dv dw dz$$

$$= N_0^2 \iiiint_{-T/2}^{T/2} k(u, v) k(w, z) [\delta(u - v)\delta(w - z) + \delta(u - w)\delta(v - z)$$

$$+ \delta(u - z)\delta(v - w)] du dv dw dz \quad \text{(by Isserlis's formula).}$$

Therefore,

$$Var\{Y | S(t) \text{ absent}\} = E\{Y^2 | S(t) \text{ absent}\} - (E\{Y | S(t) \text{ absent}\})^2$$

$$= N_0{}^2 \iiiint_{-T/2}^{T/2} k(u, v) k(w, z) [\delta(u - w)\delta(v - z) + \delta(u - z)\delta(v - w)] du dv dw dz$$

$$= 2N_0^2 \iint_{-T/2}^{T/2} k^2(u, v) du dv \quad (\text{since } k(u, v) = k(v, u)).$$

Substituting these results into the definition (11.211) of deflection yields

$$D = \frac{\left| \iint_{-T/2}^{T/2} k(u, v)[R_S(u, v) + N_0 \delta(u - v)] du dv - \iint_{-T/2}^{T/2} k(u, v) N_0 \delta(u - v) du dv \right|}{\left[2N_0^2 \iint_{-T/2}^{T/2} k^2(u, v) du dv\right]^{1/2}}$$

$$= \frac{\left| \iint_{-T/2}^{T/2} k(u, v) R_S(u, v) du dv \right|}{\left[2N_0^2 \iint_{-T/2}^{T/2} k^2(u, v) du dv\right]^{1/2}}.$$

as desired. The reason that we can assume that the kernel is symmetrical, $k(u, v) = k(v, u)$, without loss of generality is that when $k(u, v) \neq k(v, u)$, we can use $k'(u, v) \triangleq \frac{1}{2}[k(u, v) + k(v, u)]$ in place of $k(u, v)$, without affecting the detection statistic (11.209), and clearly the kernel $k'(u, v) = k'(v, u)$ is symmetrical.

11.28 *a*) Using the change of variables (11.215) in (11.214) yields the formula

$$Y = \frac{1}{N_0^2 T}\int_{-T}^{T} R_S(\tau) \int_{-(T-|\tau|)/2}^{(T-|\tau|)/2} X(t+\tau/2)X(t-\tau/2)\,dt\,d\tau = \frac{1}{N_0^2}\int_{-\infty}^{\infty} R_S(\tau)R_X(\tau)_T\,d\tau$$

for the detection statistic, for which $R_X(\tau)_T$ is the correlogram defined by (11.217).

b) Applying Parseval's relation to (11.216) and using (3.4) yields the frequency-domain formula

$$\frac{1}{N_0^2}\int_{-\infty}^{\infty} R_S(\tau)R_X(\tau)_T\,d\tau = \frac{1}{N_0^2}\int_{-\infty}^{\infty} S_S(f)\frac{1}{T}|\tilde{X}_T(f)|^2 df = \frac{1}{N_0^2}\int_{-\infty}^{\infty} S_S(f)P_T(f)\,df,$$

which is (11.218), where $P_T(f)$ is the periodogram defined by (11.219).

11.29 *a*) Substituting the optimum kernel (11.213) into the deflection formula (11.212) yields

$$D_{\max} = \frac{\left| \iint_{-T/2}^{T/2} cR_S^2(u-v)\,du\,dv \right|}{\left[2N_0^2 \iint_{-T/2}^{T/2} c^2 R_S^2(u-v)\,du\,dv\right]^{1/2}} = \left[\frac{1}{2N_0^2}\iint_{-T/2}^{T/2} R_S^2(u-v)\,du\,dv\right]^{1/2},$$

which is (11.222).

b) Using the result of exercise 7.4 in (11.222) yields the simplified formula (11.223). If T greatly exceeds the correlation time τ_0 (where $|R_S(\tau)| \ll R_S(0)$ for $\tau > \tau_0$), then

$$1 - \frac{|\tau|}{T} \simeq 1, \qquad |\tau| \le \tau_0.$$

Therefore, (11.223) can be approximated by (11.224).

11.30 Applying Parseval's relation to the detection statistic (11.226) yields

$$Y = \frac{1}{N_0}\int_{-\infty}^{\infty} h(\tau)R_X(\tau)_T\,d\tau = \frac{1}{N_0}\int_{-T}^{T}\int_{-(T-|\tau|)/2}^{(T-|\tau|)/2} h(\tau)\frac{1}{T}X(t+\tau/2)X(t-\tau/2)\,dt\,d\tau.$$

Now, by using the change of the variables $u = t + \tau/2$ and $v = t - \tau/2$, the preceding equation can be expressed as (cf. (11.214)-(11.217))

$$Y = \frac{1}{N_0 T}\iint_{-T/2}^{T/2} h(u-v)X(u)X(v)\,du\,dv = \frac{1}{N_0 T}\int_{-T/2}^{T/2} \hat{S}(u)X(u)\,du,$$

for which $\hat{S}(u)$ is given by (11.229). This is the desired estimator-correlator formula.

11.31 The $SNR_{\max}$ for a known signal in WGN is given by (11.190),

$$SNR_{\max} = \frac{E}{N_0} \simeq 2BT\frac{S_0}{N_0}.$$

The $SNR_{\max}$ for a random signal in WGN can be determined from (11.224):

$$SNR_{\max} \simeq \left[\frac{T}{2N_0^2}\int_{-\infty}^{\infty} S_S^2(f)\,df\right]^{1/2} = \left[\frac{T}{2N_0^2}2BS_0^2\right]^{1/2} = (BT)^{1/2}\frac{S_0}{N_0},$$

where S_0 is given by

$$S_0 \triangleq \left[\frac{1}{2B}\int_{-\infty}^{\infty} S_S^2(f)\,df\right]^{1/2}.$$

11.32 *a*) Multiplying both sides of the AR equation (11.230a) (with i replaced with $i+j$) by $X(i)$ and evaluating the expected value of this product yields

$$E\{X(i+j)X(i)\} + a_1E\{X(i+j-1)X(i)\} + a_2E\{X(i+j-2)X(i)\} + \cdots$$
$$+ a_nE\{X(i+j-n)X(i)\} = bE\{Z(i+j)X(i)\}$$

or, equivalently,

$$R_X(j) + a_1R_X(j-1) + a_2R_X(j-2) + \cdots + a_nR_X(j-n) = bR_{ZX}(j),$$

which is identical to (11.233).

Multiplying both sides of (11.230a) by $Z(i)$ and evaluating the expected value of this product yields

$$E\{Z(i)X(i)\} + a_1E\{Z(i)X(i-1)\} + a_2E\{Z(i)X(i-2)\} + \cdots$$
$$+ a_nE\{Z(i)X(i-n)\} = bE\{Z(i)Z(i)\}$$

or, equivalently,

$$R_{ZX}(0) + a_1R_{ZX}(1) + a_2R_{ZX}(2) + \cdots + a_nR_{ZX}(n) = bR_Z(0) = b,$$

which is identical to (11.235).

b) Substituting the regressor (11.241) into the orthogonality condition (11.242) (with p replaced by j) yields

$$E\{X(i)X(i-j)\} = R_X(j) = E\{\hat{X}(i)X(i-j)\}$$
$$= \sum_{p=1}^{n}(-a_p)E\{X(i-p)X(i-j)\} = -\sum_{p=1}^{n}a_pR_X(j-p), \qquad j \ge 1,$$

which is (11.236a), as desired.

c) For $n = 2$, we obtain from the Levinson-Durbin algorithm (11.237) that

$$a_2(2) = \frac{-1}{b^2(1)}[R_X(2) + a_1(1)R_X(1)]$$

$$= \frac{-R_X(0)}{[R_X(0)]^2 - [R_X(1)]^2}\big[R_X(2) - [R_X(1)]^2/R_X(0)\big]$$

$$= \frac{[R_X(1)]^2 - R_X(0)R_X(2)}{[R_X(0)]^2 - [R_X(1)]^2}, \qquad (*)$$

$$b^2(2) = [1 - a_2^2(2)]b^2(1)$$

$$= \left\{1 - [\frac{[R_X(1)]^2 - R_X(0)R_X(2)}{[R_X(0)]^2 - [R_X(1)]^2}]^2\right\} \frac{[R_X(0)]^2 - [R_X(1)]^2}{R_X(0)}$$

$$= \frac{[R_X(0)]^3 - 2R_X(0)[R_X(1)]^2 - R_X(0)[R_X(2)]^2 + 2[R_X(1)]^2R_X(2)}{[R_X(0)]^2 - [R_X(1)]^2}, \qquad (***)$$

and

$$a_1(2) = a_1(1) + a_2(2)a_1(1) = -\frac{R_X(1)}{R_X(0)}[1 + \frac{[R_X(1)]^2 - R_X(0)R_X(2)}{[R_X(0)]^2 - [R_X(1)]^2}]$$

$$= \frac{R_X(1)R_X(2) - R_X(1)R_X(0)}{[R_X(0)]^2 - [R_X(1)]^2}. \qquad (**)$$

Alternatively, we can expand the Yule-Walker equations (11.236a) with $n = 2$ and j=1, 2, to obtain

$$R_X(1) = -a_1R_X(0) - a_2R_X(1)$$

$$R_X(2) = -a_1R_X(1) - a_2R_X(0),$$

and then solve them for a_1 and a_2 to obtain

$$a_1 = \frac{R_X(1)R_X(2) - R_X(1)R_X(0)}{[R_X(0)]^2 - [R_X(1)]^2}$$

$$a_2 = \frac{[R_X(1)]^2 - R_X(2)R_X(0)}{[R_X(0)]^2 - [R_X(1)]^2},$$

which agree with (**) and (*). Then, we can solve (11.236b) for b^2 to obtain

$$b^2 = R_X(0) + a_1R_X(1) + a_2R_X(2)$$

$$= R_X(0) + R_X(1)\frac{R_X(1)R_X(2) - R_X(1)R_X(0)}{[R_X(0)]^2 - [R_X(1)]^2} + R_X(2)\frac{[R_X(1)]^2 - R_X(2)R_X(0)}{[R_X(0)]^2 - [R_X(1)]^2}$$

$$= \frac{[R_X(0)]^3 - 2R_X(0)[R_X(1)]^2 - R_X(0)[R_X(2)]^2 + 2[R_X(1)]^2R_X(2)}{[R_X(0)]^2 - [R_X(1)]^2},$$

which agrees with (***). Therefore, the Levinson-Durbin algorithm yields the solution to the Yule-Walker equations.

11.33 Substituting (11.240) and (11.241) into the necessary and sufficient orthogonality condition (11.242) yields (11.283). Replacing i with $i-p$ in (11.240) and then multi%plying both sides of (11.240) by $Z(i)$ and evaluating the expected value yields

$$E\{Z(i)Z(i-p)\} = \frac{1}{b(n)}[E\{Z(i)X(i-p) - \sum_{q=1}^{n}[-a_q(n)]E\{Z(i)X(i-p-q)\}]$$

$$= \frac{1}{b(n)}\sum_{q=1}^{n} a_q(n)E\{Z(i)X(i-p-q)\}, \quad p = 1, 2, 3, \ldots n \quad \text{(using (11.283))},$$

which is (11.284). Further use of (11.283) in (11.284) leads to (11.285). Thus, since $b(\cdot)$ and $R_{ZX}(\cdot)$ are bounded sequences (as explained in the exercise statement), and the number of terms in the summation in (11.285) is finite, and $a_q(n) \to 0$ as $n \to \infty$, then $R_Z(p) = 0$ for all $p \neq 0$ in the limit $n \to \infty$. Thus, the prediction error $Z(i)$ becomes white as the order n of the predictor grows without bound.

11.34 *a*) For $\mathbf{Y}_n \triangleq \mathbf{AX}_n$, we obtain from (1.41)

$$f_{\mathbf{Y}_n}(\mathbf{Y}_n) = \frac{1}{|\mathbf{A}|} f_{\mathbf{X}_n}(\mathbf{X}_n)$$

and, therefore, the relative entropy of $\mathbf{Y}_n$ is given by

$$H(\mathbf{Y}_n) = E\{\ln[\frac{1}{f_{\mathbf{Y}_n}(\mathbf{Y}_n)}]\} = E\{\ln[\frac{1}{f_{\mathbf{X}_n}(\mathbf{X}_n)}]\} + \ln(|\mathbf{A}|)$$

$$= H(\mathbf{X}_n) + \ln(|\mathbf{A}|).$$

b) For a lower triangular matrix, the determinant equals the product of diagonal elements. Thus,

$$|\mathbf{A}| = g^n(0).$$

Hence, we have

$$\ln(|\mathbf{A}|) = n \ln|g(0)|.$$

c) From (11.247), the relative entropy rate for $Y(i)$ is given by

$$\overline{H}_Y = \lim_{n\to\infty} \frac{1}{n} E\{\ln[\frac{1}{f_{\mathbf{Y}_n}(\mathbf{Y}_n)}]\} = \lim_{n\to\infty} \frac{1}{n} H(\mathbf{X}_n) + \ln|g(0)| = \overline{H}_X + \ln|g(0)|,$$

which is (11.288).

d) Finally, substituting (11.289) into (11.288) yields the difference

$$\overline{H}_Y - \overline{H}_X = \ln g(0)| = \int_{-1/2}^{1/2} \ln|G(f)|df = \frac{1}{2}\int_{-1/2}^{1/2} \ln|G(f)|^2 df,$$

which is the desired result (11.249).

11.35 To derive the backward-prediction-error recursion (11.258), we proceed as follows. Substituting (11.237e) into (11.258d) yields

$$\check{E}_n(i) = X(i-n) + \sum_{p=1}^{n-1} [a_p(n-1) + a_n(n)a_{n-p}(n-1)]X(i-n+p) + a_n(n)X(i)$$

$$= X(i-n) + \sum_{p=1}^{n-1} a_p(n-1)X(i-n+p) + a_n(n)\sum_{p=1}^{n-1} a_{n-p}(n-1)X(i-n+p)$$

$$+ a_n(n)X(i)$$

$$= \check{E}_{n-1}(i-1) + a_n(n)[\sum_{q=1}^{n-1} a_q(n-1)X(i-q) + X(i)]$$

$$= \check{E}_{n-1}(i-1) + a_n(n)\check{E}_{n-1}(i),$$

which is the desired result (11.258e).

11.36 It follows from the orthogonality condition (11.242) that

$$E\{[X(i) - \hat{X}(i)]\hat{X}(i)\} = 0$$

and, therefore, that the minimum mean-squared error β^2 for mth-order predictor for any stationary process is given by

$$\beta^2 = E\{X^2(i)\} - E\{\hat{X}(i)X(i)\}. \qquad (*)$$

Substituting the expression

$$\hat{X}(i) = \sum_{p=1}^{m} (-\alpha_p)X(i-p) \qquad (**)$$

for the mth-order predictor into (*) yields

$$R_X(0) = \beta^2 - \sum_{p=1}^{m} \alpha_p R_X(-p).$$

Also, substituting (**) into (11.242) with p there replaced by j yields

$$R_X(j) = -\sum_{p=1}^{m} \alpha_p R_X(j-p), \quad j \geq 1.$$

These last two equations are identical in form to the Yule-Walker equations (11.236). Now, if $X(i)$ is an nth-order AR process (11.230a), then it can be treated as an mth-order AR process $(m > n)$ with parameters $a_p = 0$ for $p > n$. It therefore follows from the Yule-Walker equations (11.236) that the above equations are satisfied with

$$\begin{aligned} \alpha_p &= a_p, & 1 \le p \le n \\ \alpha_p &= 0, & p > n \\ \beta &= b. \end{aligned}$$

Thus, the mean-squared prediction error for order $m > n$ is no smaller than that for order $m = n$.

11.37 *a*) By using the Levinson-Durbin algorithm (11.237) for the specified process, we obtain

$$a_1(1) = -R_X(1)/R_X(0) = -r$$

$$b^2(1) = [1 - a_1^2(1)]R_X(0) = 1 - r^2$$

$$a_2(2) = \frac{-1}{b^2(1)}[R_X(2) + \sum_{q=1}^{1} a_q(1)R_X(2-q)] = \frac{-1}{1-r^2}[r^2 - r^2] = 0$$

$$b^2(2) = [1 - a_2^2(2)]b^2(1) = b^2(1) = 1 - r^2$$

$$a_1(2) = a_1(1) + a_2(2)a_1(1) = a_1(1) = -r$$

$$a_3(3) = \frac{-1}{b^2(2)}[R_X(3) + \sum_{q=1}^{2} a_q(2)R_X(3-q)] = \frac{-1}{1-r^2}[r^3 - r^3] = 0$$

$$b^2(3) = [1 - a_3^2(3)]b^2(2) = b^2(2) = b^2(1) = 1 - r^2$$

$$a_2(3) = a_2(2) + a_3(3)a_1(2) = 0$$

$$a_1(3) = a_1(2) + a_3(3)a_2(2) = a_1(2) = -r$$

$$\vdots$$

$$a_n(n) = a_{n-1}(n) = \cdots = a_2(n) = 0$$

$$b^2(n) = b^2(n-1) = \cdots = b^2(1) = 1 - r^2$$

$$a_1(n) = a_1(n-1) = \cdots = a_1(1) = -r.$$

Thus, the nth-order predictor is identical to the first-order predictor for this process:

$$\hat{X}(i) = rX(i-1).$$

Therefore, for $r > 0$, $\hat{X}(i)$ is an attenuated version of $X(i-1)$ with the same sign, and for $r < 0$, $\hat{X}(i)$ is an attenuated version of $X(i-1)$ but with opposite sign, reflecting the fact that $X(i)$ is an oscillatory process for $r < 0$.

b) The autocorrelation for the specified process $X(t)$ is given by

$$R_X(i) = N_0 h(i) \otimes h(i),$$

where N_0 is the average power of the white excitation. We can use the Levinson-Durbin algorithm (11.237) to obtain coefficients $\{a_j(n)\}_{j=1}^{n}$ for the nth-order predictor. The result for any $n > 2$ is

$$a_n(n) = a_{n-1}(n) = \cdots = a_3(n) = 0$$

$$b^2(n) = b^2(n-1) = \cdots = b^2(2).$$

That is, an nth-order predictor can do no better than a 2nd-order predictor. The reason for this is that $X(i)$ is 2nd-order ARMA process, which is a 2nd-order Markov process. In comparison, $X(i)$ in part a is a 1st-order AR process, which is a 1st-order Markov process.

Chapter 12

Cyclostationary Processes

12.1 To verify the Fourier-coefficient formula (12.6), we substitute (12.4) into (12.6) (with α replaced by β) to obtain

$$R_X^\beta(\tau) = \lim_{Z\to\infty} \frac{1}{Z} \int_{-Z/2}^{Z/2} R_X(t+\tau/2,\, t-\tau/2) e^{-i2\pi\beta t}\, dt$$

$$= \sum_\alpha R_X^\alpha(\tau) \lim_{Z\to\infty} \frac{1}{Z} \int_{-Z/2}^{Z/2} e^{i2\pi\alpha t} e^{-i2\pi\beta t}\, dt$$

$$= \sum_\alpha R_X^\alpha(\tau) \lim_{Z\to\infty} \frac{\sin[\pi(\alpha-\beta)Z]}{\pi(\alpha-\beta)Z} = \begin{cases} R_X^\beta(\tau), & \alpha=\beta \\ 0, & \alpha\neq\beta. \end{cases}$$

Therefore, (12.6) is indeed the correct formula for the Fourier-coefficient.

12.2 *a*) Since

$$R_Y(t+\tau/2,\, t-\tau/2) = E\{Y(t+\tau/2)Y(t-\tau/2)\} = E\{X(t+t_0+\tau/2)X(t+t_0-\tau/2)\}$$
$$= R_X(t+t_0+\tau/2,\, t+t_0-\tau/2),$$

then the cyclic autocorrelation for the translated process $Y(t)$ is

$$R_Y^\alpha(\tau) = \lim_{Z\to\infty} \frac{1}{Z} \int_{-Z/2}^{Z/2} R_Y(t+\tau/2,\, t-\tau/2) e^{-i2\pi\alpha t}\, dt$$

$$= \lim_{Z\to\infty} \frac{1}{Z} \int_{-Z/2}^{Z/2} R_X(t+t_0+\tau/2,\, t+t_0-\tau/2) e^{-i2\pi\alpha t}\, dt$$

$$= \lim_{Z\to\infty} \frac{1}{Z} \int_{-Z/2+t_0}^{Z/2+t_0} R_X(t'+\tau/2,\, t'-\tau/2) e^{-i2\pi\alpha t'} dt' e^{i2\pi\alpha t_0} \qquad (t' = t + t_0)$$

$$= R_X^\alpha(\tau) e^{i2\pi\alpha t_0}.$$

b) For a real vector process $\mathbf{X}(t)$, the cyclic autocorrelation matrix is given by

$$\mathbf{R}_\mathbf{X}^\alpha(\tau) = \langle E\{\mathbf{X}(t+\tau/2)\mathbf{X}^T(t-\tau/2)\} e^{-i2\pi\alpha t}\rangle = \langle \mathbf{R}_\mathbf{X}(t+\tau/2,\, t-\tau/2) e^{-i2\pi\alpha t}\rangle.$$

It therefore follows that $\mathbf{R}_\mathbf{X}^\alpha(\tau)$ has the symmetry properties

$$\mathbf{R}_{\mathbf{X}}^{\alpha}(-\tau) = \langle E\{\mathbf{X}(t-\tau/2)\mathbf{X}^T(t+\tau/2)\}e^{-i2\pi\alpha t}\}\rangle$$
$$= \langle \mathbf{R}_{\mathbf{X}}(t+\tau/2,\ t-\tau/2)^T e^{-i2\pi\alpha t}\rangle \ = \mathbf{R}_{\mathbf{X}}^{\alpha}(\tau)^T$$

and

$$\mathbf{R}_{\mathbf{X}}^{-\alpha}(\tau) = \langle \mathbf{R}_{\mathbf{X}}(t+\tau/2,\ t-\tau/2)e^{i2\pi\alpha t}\rangle$$
$$= \langle \mathbf{R}_{\mathbf{X}}(t+\tau/2,\ t-\tau/2)e^{-i2\pi\alpha t}\rangle^* \ = \mathbf{R}_{\mathbf{X}}^{\alpha}(\tau)^* .$$

12.3 *a*) The time-averaged autocorrelation of the frequency-translated process $U(t)$ is given by

$$\langle R_U\rangle(\tau) = \langle E\{U(t+\tau/2)U(t-\tau/2)^*\rangle$$
$$= \langle E\{X(t+\tau/2)e^{-i\pi\alpha(t+\tau/2)}X(t-\tau/2)e^{i\pi\alpha(t-\tau/2)}\}\rangle$$
$$= \langle E\{X(t+\tau/2)X(t-\tau/2)\}\rangle e^{-i\pi\alpha\tau} = \langle R_X\rangle(\tau)e^{-i\pi\alpha\tau}$$

and similarly for $V(t)$. The time-averaged cross-correlation of the two frequency-translated processes $U(t)$ and $V(t)$ is given by

$$\langle R_{UV}\rangle(\tau) = \langle E\{U(t+\tau/2)V(t-\tau/2)^*\}\rangle$$
$$= \langle E\{X(t+\tau/2)e^{-i\pi\alpha(t+\tau/2)}X(t-\tau/2)e^{-i\pi\alpha(t-\tau/2)}\}\rangle$$
$$= \langle E\{X(t+\tau/2)X(t-\tau/2)e^{-i2\pi\alpha t}\}\rangle = R_X^{\alpha}(\tau).$$

b) Fourier transforming (12.296a) and (12.296b) yields the symmetry properties

$$\int_{-\infty}^{\infty}\mathbf{R}_{\mathbf{X}}^{\alpha}(-\tau)e^{-i2\pi f\tau}d\tau = \int_{-\infty}^{\infty}\mathbf{R}_{\mathbf{X}}^{\alpha}(\tau)e^{-i2\pi(-f)\tau}d\tau = \mathbf{S}_{\mathbf{X}}^{\alpha}(-f)$$
$$= \int_{-\infty}^{\infty}\mathbf{R}_{\mathbf{X}}^{\alpha}(\tau)^T e^{-i2\pi f\tau}d\tau = \mathbf{S}_{\mathbf{X}}^{\alpha}(f)^T ,$$

which is (12.297a), and

$$\int_{-\infty}^{\infty}\mathbf{R}_{\mathbf{X}}^{\alpha}(\tau)^T e^{-i2\pi f\tau}d\tau = \mathbf{S}_{\mathbf{X}}^{-\alpha}(f)^T \ = \int_{-\infty}^{\infty}\mathbf{R}_{\mathbf{X}}^{\alpha}(\tau)^* e^{-i2\pi f\tau}d\tau$$
$$= \Big[\int_{-\infty}^{\infty}\mathbf{R}_{\mathbf{X}}(\tau)e^{-(-i)2\pi f\tau}d\tau\Big]^* \ = \mathbf{S}_{\mathbf{X}}^{\alpha}(f)^* ,$$

which is (12.297b).

12.4 Let $h(t)$ denote the window function

$$h(t) = \begin{cases} 1, & |t| < W/2 \\ 0, & otherwise ; \end{cases}$$

the cyclic-correlogram (12.25) can then be reexpressed as

$$R_X^\alpha(t, \tau)_W = \frac{1}{W}\int_{-\infty}^{\infty} X(t+u+\tau/2)X(t+u-\tau/2)h(u+\tau/2)h(u-\tau/2)e^{-i2\pi\alpha(t+u)}du$$

$$= \frac{1}{W}\int_{-\infty}^{\infty} [X(t+v+\tau)h(v+\tau)e^{-i\pi\alpha(t+v+\tau)}][X(t+v)h(v)e^{-i\pi\alpha(t+v)}]dv \quad (v = u - \tau/2)$$

$$= \frac{1}{W}\int_{-\infty}^{\infty} Y(t, v+\tau)Y(t, v)dv,$$

where the two bracketed terms have been identified as $Y(t, v+\tau)$ and $Y(t, v)$. Fourier transforming and applying the convolution theorem yields

$$\int_{-\infty}^{\infty} R_X^\alpha(t, \tau)_W e^{-i2\pi f\tau}d\tau = \frac{1}{W}\tilde{Y}(t, f)\tilde{Y}(t, -f).$$

But

$$\tilde{Y}(t, f) = \int_{-\infty}^{\infty} X(t+v)h(v)e^{-i2\pi(f+\alpha/2)(t+v)}dv = \int_{-W/2}^{W/2} X(t+v)e^{-i2\pi(f+\alpha/2)(t+v)}dv$$

$$= \int_{t-W/2}^{t+W/2} X(v)e^{-i2\pi(f+\alpha/2)v}\,dv = \tilde{X}_W(t, f+\alpha/2) \quad \text{(using (12.20))}$$

and

$$\tilde{Y}(t, -f) = \tilde{X}_W(t, -f+\alpha/2) = X_W(t, f-\alpha/2)^*.$$

Thus,

$$\int_{-\infty}^{\infty} R_X^\alpha(t, \tau)_W e^{-i2\pi f\tau}d\tau = \frac{1}{W}\tilde{X}_W(t, f+\alpha/2)\tilde{X}_W(t, f-\alpha/2)^* = P_X^\alpha(t, f)_W$$

(using (12.24)). This verifies the cyclic-periodogram-cyclic-correlogram relation.

12.5 *a*) To obtain (12.41), we substitute (12.4) into (12.39) to obtain

$$\langle R_X\rangle_T(t, u) = \lim_{N\to\infty}\frac{1}{2N+1}\sum_{n=-N}^{N} R_X(t+nT, u+nT)$$

$$= \lim_{N\to\infty}\frac{1}{2N+1}\sum_{n=-N}^{N}\sum_{\alpha} R_X^\alpha(t-u)e^{i\pi\alpha(t+nT+u+nT)}$$

$$= \sum_{\alpha} R_X^\alpha(t-u)e^{i\pi\alpha(t+u)}\lim_{N\to\infty}\frac{1}{2N+1}\sum_{n=-N}^{N} e^{i2\pi\alpha nT}.$$

Let us consider the quantity

$$J(\alpha T) \triangleq \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} e^{i2\pi\alpha nT}$$

$$= \lim_{N\to\infty} \frac{1}{2N+1}\Big[\frac{1-e^{i2\pi\alpha(N+1)T}}{1-e^{i2\pi\alpha T}} + \frac{1-e^{-i2\pi\alpha(N+1)T}}{1-e^{-i2\pi\alpha T}} - 1\Big]$$

$$= \lim_{N\to\infty} \frac{1}{2N+1}\Big[2\mathrm{Re}\Big\{\frac{1-e^{i2\pi\alpha(N+1)T}}{1-e^{i2\pi\alpha T}}\Big\} - 1\Big]$$

$$= \lim_{N\to\infty} \frac{1}{2N+1}\Big[2\cos(\pi\alpha NT)\frac{\sin[\pi\alpha(N+1)T]}{\sin(\pi\alpha T)} - 1\Big].$$

It is easily seen that for $\alpha T \neq \textit{integer}$, $J(\alpha T) = 0$. For $\alpha T = \textit{integer}$, we have

$$J(\alpha T) = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} 1 = 1.$$

Therefore,

$$<R_X>_T(t, u) = \sum_m R_X^{m/T}(t-u)e^{i\pi m(t+u)/T},$$

where m represents all the integer values of αT.

b) Substituting (12.41) into (12.43), with T replaced by T_j, yields

$$R_X(t, u) = <R_X>(t-u) + \sum_j \Big[\sum_{m=-\infty}^{\infty} R_X^{m/T_j}(t-u)e^{i\pi m(t+u)/T_j} - <R_X>(t-u)\Big].$$

The fact that (12.6) with $\alpha = 0$ is equivalent to (12.37), that is,

$$R_X^0(t-u) = \lim_{Z\to\infty} \frac{1}{Z}\int_{-Z/2}^{Z/2} R_X(v+t, v+u)dv = <R_X>(t-u),$$

can be used to further reduce (12.43) as follows:

$$R_X(t, u) = R_X^0(t-u) + \sum_j \sum_{\substack{m=-\infty \\ m\neq 0}}^{\infty} R_X^{m/T_j}(t-u)e^{i\pi m(t+u)/T_j}.$$

But this equation is equivalent to (12.4), which can be expressed as

$$R_X(t, u) = \sum_\alpha R_X^\alpha(t-u)e^{i\pi\alpha(t+u)} = R_X^0(t-u) + \sum_{\alpha\neq 0} R_X^\alpha(t-u)e^{i\pi\alpha(t+u)},$$

where α represents all integer multiples of all the fundamental frequencies $\{1/T_j\}$.

12.6 *a*) The input-output cyclic-autocorrelation relation for linear periodically time-variant transformations can be derived as follows. Applying the definition (12.6) to (12.44) with (12.45)-(12.46) substituted in yields

$$R_Y^\alpha(\tau) = \lim_{Z\to\infty} \frac{1}{Z}\int_{-Z/2}^{Z/2} R_Y(t+\tau/2, t-\tau/2)e^{-i2\pi\alpha t}\,dt$$

$$= \lim_{Z\to\infty} \frac{1}{Z}\int_{-Z/2}^{Z/2}\iint_{-\infty}^{\infty} E\{\mathbf{h}(t+\tau/2,\, u)\mathbf{X}(u)[\mathbf{X}^T(v)\mathbf{h}^T(t-\tau/2,\, v)]^*\}dudv\ e^{-i2\pi\alpha t}\,dt$$

$$= \lim_{Z\to\infty} \frac{1}{Z}\int_{-Z/2}^{Z/2}\iint_{-\infty}^{\infty} \mathbf{h}(t+\tau/2,\, u)\mathbf{R}_{\mathbf{X}}(u,\, v)\mathbf{h}^T(t-\tau/2,\, v)^*\,dudv\ e^{-i2\pi\alpha t}\,dt$$

$$= \lim_{Z\to\infty} \frac{1}{Z}\int_{-Z/2}^{Z/2}\iint_{-\infty}^{\infty} \sum_{n,m=-\infty}^{\infty} \mathbf{g}_n(t+\tau/2-u)\mathbf{R}_{\mathbf{X}}(u,\, v)\mathbf{g}_m^T(t-\tau/2-v)^*$$

$$\times\, e^{i2\pi(nu-mv)/T}\,dudv\ e^{-i2\pi\alpha t}\,dt$$

$$= \lim_{Z\to\infty} \frac{1}{Z}\int_{-Z/2}^{Z/2}\iint_{-\infty}^{\infty} \sum_{n,m=-\infty}^{\infty} \mathbf{g}_n(t+\tau/2-u)\sum_{\beta}\mathbf{R}_{\mathbf{X}}^{\beta}(u-v)e^{i\pi\beta(u+v)}\mathbf{g}_m^T(t-\tau/2-v)^*$$

$$\times\, e^{i2\pi(nu-mv)/T}\,dudv e^{-i2\pi\alpha t}\,dt.$$

Using the changes of variables $\tau' = v - u$ and $t' = t - (u+v)/2$ in the above equation, and then time-averaging yields (12.298). Using the given matrix identities in (12.298) then yields (12.299), from which the desired result (12.50) follows.

b) Since

$$\int_{-\infty}^{\infty} \mathbf{r}_{nm}^{\alpha}(-\tau)e^{-i2\pi f\tau}d\tau = \mathbf{G}_n^T(-f+\alpha/2)\mathbf{G}_m^*(-f-\alpha/2)$$

and

$$\int_{-\infty}^{\infty} \mathbf{R}_{\mathbf{X}}^{\alpha-(n-m)/T}(\tau)e^{-i\pi(n+m)\tau/T}e^{-i2\pi f\tau}d\tau = \mathbf{S}_{\mathbf{X}}^{\alpha-(n-m)/T}(f+\frac{n+m}{2T}),$$

then applying the convolution theorem to (12.50) yields

$$S_Y^{\alpha}(f) = \int_{-\infty}^{\infty} R_Y^{\alpha}(\tau)e^{-i2\pi f\tau}d\tau$$

$$= \sum_{n,m=-\infty}^{\infty} tr\,\{\mathbf{S}_{\mathbf{X}}^{\alpha-(n-m)/T}(f+\frac{n+m}{2T})\mathbf{G}_n^T(-f+\alpha/2)\mathbf{G}_m^*(-f-\alpha/2)\}$$

$$= \sum_{n,m=-\infty}^{\infty} \mathbf{G}_n(-f+\alpha/2)[\mathbf{S}_{\mathbf{X}}^{\alpha-(n-m)/T}(f+\frac{n+m}{2T})]^T\,\mathbf{G}_m^T(-f-\alpha/2)^*$$

$$= \sum_{n,m=-\infty}^{\infty} \mathbf{G}_n(-f+\alpha/2)\mathbf{S}_{\mathbf{X}}^{\alpha-(n-m)/T}(-f-\frac{n+m}{2T})\mathbf{G}_m^T(-f-\alpha/2)^*$$

$$= \sum_{n,m=-\infty}^{\infty} \mathbf{G}_n(f+\alpha/2)\mathbf{S}_{\mathbf{X}}^{\alpha-(n-m)/T}(f-\frac{n+m}{2T})\mathbf{G}_m^T(f-\alpha/2)^*$$

(since $S_Y^\alpha(f) = S_Y^\alpha(-f)$), which is the desired input-output cyclic-spectrum relation (12.53).

c) For a linear time-invariant transformation we have $\mathbf{h}(t, u) = \mathbf{h}(t-u)$ and, therefore, (12.46) yields

$$\mathbf{g}_n(\tau) = \begin{cases} \mathbf{h}(\tau), & n = 0 \\ 0, & n \neq 0. \end{cases}$$

Consequently,

$$\mathbf{r}_{nm}^\alpha(\tau) = \int_{-\infty}^{\infty} \mathbf{g}_n^T(t+\tau/2)\mathbf{g}_m^*(t-\tau/2)e^{-i2\pi\alpha t}\,dt = \mathbf{r}_{\mathbf{h}}^\alpha(\tau)\delta_n\delta_m.$$

Therefore, the input-output relation (12.50) reduces to

$$R_Y^\alpha(\tau) = tr\,\{\mathbf{R}_{\mathbf{X}}^\alpha(\tau) \otimes \mathbf{r}_{\mathbf{h}}^\alpha(-\tau)\},$$

which is (12.55) and, as a result,

$$S_Y^\alpha(f) = \int_{-\infty}^{\infty} R_Y^\alpha(\tau)e^{-i2\pi f\tau}d\tau = tr\,\{\mathbf{S}_{\mathbf{X}}^\alpha(f)\mathbf{H}^T(-f+\alpha/2)\mathbf{H}^*(-f-\alpha/2)\}$$

$$= \mathbf{H}(-f+\alpha/2)\mathbf{S}_{\mathbf{X}}^\alpha(-f)\mathbf{H}^T(-f-\alpha/2)^* = \mathbf{H}(f+\alpha/2)\mathbf{S}_{\mathbf{X}}^\alpha(f)\mathbf{H}^T(f-\alpha/2)^*$$

(using $S_Y^\alpha(f) = S_Y^\alpha(-f)$), which is (12.56).

d) Let $p = n - m$; (12.50) can then be expressed as

$$R_Y^\alpha(\tau) = \sum_{n,p=-\infty}^{\infty} tr\,\{[\mathbf{R}_{\mathbf{X}}^{\alpha-p/T}(\tau)e^{-i\pi(2n-p)/T}] \otimes \mathbf{r}_{n(n-p)}^\alpha(-\tau)\}.$$

If $X(t)$ exhibits no cyclostationarity, then

$$\mathbf{R}_{\mathbf{X}}^{\alpha-p/T}(\tau) = \begin{cases} \langle\mathbf{R}_{\mathbf{X}}\rangle(\tau), & \alpha = p/T \\ 0, & \alpha \neq p/T. \end{cases}$$

Therefore, the input-output relation (12.50) reduces to

$$R_Y^\alpha(\tau) = \begin{cases} \sum\limits_{n=-\infty}^{\infty} tr\,\{[\langle\mathbf{R}_{\mathbf{X}}\rangle(\tau)e^{-i\pi(2n-p)/T}] \otimes \mathbf{r}_{n(n-p)}^\alpha(-\tau)\}, & \alpha = p/T \\ 0, & \alpha \neq p/T, \end{cases}$$

which is (12.58), and its Fourier transform (12.53) reduces to

$$S_Y^\alpha(f) = \begin{cases} \sum\limits_{n=-\infty}^{\infty} \mathbf{G}_n(f+\frac{p}{2T})\langle\mathbf{S}_{\mathbf{X}}\rangle(f-\frac{2n-p}{2T})\mathbf{G}_{n-p}^T(f-\frac{p}{2T})^*, & \alpha = p/T \\ 0, & \alpha \neq p/T \end{cases}$$

$$= \begin{cases} \sum_{m=-\infty}^{\infty} \mathbf{G}_{m+p}(f+\alpha/2) \langle \mathbf{S_X} \rangle (f - \frac{\alpha}{2} - \frac{m}{T}) \mathbf{G}_m^T(f-\alpha/2), & \alpha = p/T \\ 0 & \alpha \neq p/T, \end{cases}$$

which is (12.59).

12.7 *a*) To derive the relation (12.61), substitute (12.44)-(12.46) into (12.60) to obtain

$$\mathbf{R}_{\mathbf{X}Y}^{\alpha}(\tau) = \lim_{Z\to\infty} \frac{1}{Z} \int_{-Z/2}^{Z/2} E\{\mathbf{X}(t+\tau/2) \int_{-\infty}^{\infty} \mathbf{X}^T(u)\mathbf{h}^T(t-\tau/2, u)du\} e^{-i2\pi\alpha t} dt$$

$$= \lim_{Z\to\infty} \frac{1}{Z} \int_{-Z/2}^{Z/2} \int_{-\infty}^{\infty} \mathbf{R_X}(t+\tau/2, u)\mathbf{h}^T(t-\tau/2, u) du e^{-i2\pi\alpha t} dt$$

$$= \lim_{Z\to\infty} \frac{1}{Z} \int_{-Z/2}^{Z/2} \int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \mathbf{R}_X(t+\tau/2, u)\mathbf{g}_n^T(t-\tau/2-u) e^{i2\pi n u/T} du e^{-i2\pi\alpha t} dt$$

$$= \lim_{Z\to\infty} \frac{1}{Z} \int_{-Z/2}^{Z/2} \int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \mathbf{R_X}(t+\tau/2, t-\tau/2-w)\mathbf{g}_n^T(w)$$

$$\times e^{-i2\pi(\alpha-n/T)t} e^{-i2\pi n(w+\tau/2)/T} dw dt \qquad (\text{using } w = t - \tau/2 - u)$$

$$= \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} \mathbf{R}_{\mathbf{X}}^{\alpha-n/T}(\tau+w) e^{-i\pi(\alpha-n/T)w} \mathbf{g}_n^T(w) e^{-i2\pi n(w+\tau/2)/T} dw$$

$$= \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} \mathbf{R}_{\mathbf{X}}^{\alpha-n/T}(\tau+w) e^{-i\pi n(w+\tau)/T} \mathbf{g}_n^T(w) e^{-i\pi\alpha w} dw$$

$$= \sum_{n=-\infty}^{\infty} [\mathbf{R}_{\mathbf{X}}^{\alpha-n/T}(\tau) e^{-i\pi n\tau/T}] \otimes [\mathbf{g}_n^T(-\tau) e^{i\pi\alpha\tau}]$$

$$= \sum_{m=-\infty}^{\infty} [\mathbf{R}_{\mathbf{X}}^{\alpha+m/T}(\tau) e^{i\pi m\tau/T}] \otimes [\mathbf{g}_m^T(-\tau)^* e^{i\pi\alpha\tau}] \quad (\text{since } \mathbf{g}_{-n} = \mathbf{g}_n^*).$$

b) Applying the convolution theorem to (12.61) yields

$$\mathbf{S}_{\mathbf{X}Y}^{\alpha}(f) = \int_{-\infty}^{\infty} \mathbf{R}_{\mathbf{X}Y}^{\alpha}(\tau) e^{-i2\pi f\tau} d\tau = \sum_{m=-\infty}^{\infty} \mathbf{S}_{\mathbf{X}}^{\alpha+m/T}(f - \frac{m}{2T}) \mathbf{G}_m^T(f-\alpha/2)^*,$$

as desired.

c) For a linear time-invariant transformation, we have $\mathbf{h}(t, u) = \mathbf{h}(t-u)$ and, therefore,

$$\mathbf{g}_m(\tau) = \begin{cases} \mathbf{h}(\tau), & m = 0 \\ 0, & m \neq 0. \end{cases}$$

Consequently, (12.61) and (12.62) reduce to

$$\mathbf{R}_{\mathbf{X}Y}^{\alpha}(\tau) = \mathbf{R}_{\mathbf{X}}^{\alpha}(\tau) \otimes [\mathbf{h}^T(-\tau)^* e^{i\pi\alpha\tau}]$$

and

$$\mathbf{S}_{\mathbf{X}Y}^{\alpha}(f) = \mathbf{S}_{\mathbf{X}}^{\alpha}(f)\mathbf{H}^T(f - \alpha/2)^*,$$

as desired.

d) If $X(t)$ exhibits no cyclostationarity, then

$$\mathbf{R}_{\mathbf{X}}^{\alpha+m/T}(\tau) = \begin{cases} \langle\mathbf{R}_{\mathbf{X}}\rangle(\tau), & \alpha = -m/T \\ 0, & \alpha \neq -m/T. \end{cases}$$

Therefore, (12.61) and (12.62) reduce to

$$\mathbf{R}_{\mathbf{X}Y}^{\alpha}(\tau) = \begin{cases} [\langle\mathbf{R}_{\mathbf{X}}\rangle(\tau)e^{-i\pi\alpha\tau}] \otimes [\mathbf{g}_{-p}^T(-\tau)^* e^{i\pi\alpha\tau}], & \alpha = p/T \\ 0, & \alpha \neq p/T \end{cases}$$

$$= \begin{cases} [\langle\mathbf{R}_{\mathbf{X}}\rangle(\tau)e^{-i\pi\alpha\tau}] \otimes [\mathbf{g}_{p}^T(-\tau)e^{i\pi\alpha\tau}], & \alpha = p/T \\ 0, & \alpha \neq p/T \end{cases}$$

and

$$\mathbf{S}_{\mathbf{X}Y}^{\alpha}(f) = \begin{cases} \langle\mathbf{S}_{\mathbf{X}}\rangle(f + \alpha/2)\mathbf{G}_{-p}^T(f - \alpha/2)^*, & \alpha = p/T \\ 0, & \alpha \neq p/T \end{cases}$$

$$= \begin{cases} \langle\mathbf{S}_{\mathbf{X}}\rangle(f + \alpha/2)\mathbf{G}_{p}^T(f - \alpha/2), & \alpha = p/T \\ 0, & \alpha \neq p/T \end{cases}$$

for all integers p, as desired.

12.8 *a*) Let $\mathbf{X}(t) = [Z(t), \quad W(t)]^T$ and $\mathbf{h}(t, u)$ be given by (12.77); we then obtain the QAM form from (12.44):

$$Q(t) = \int_{-\infty}^{\infty} \mathbf{h}(t, u)\mathbf{X}(u)du$$

$$= \int_{-\infty}^{\infty} [\cos(2\pi f_0 t)\delta(t-u)Z(u) + \sin(2\pi f_0 t)\delta(t-u)W(u)]du$$

$$= Z(t)\cos(2\pi f_0 t) + W(t)\sin(2\pi f_0 t),$$

which is (12.76).

b) To obtain the cyclic autocorrelation for QAM, we proceed as follows. Using (12.77) and (12.46), we obtain

$$\mathbf{g}_n(\tau) = \begin{cases} \frac{1}{2}\delta(\tau)[\,1, \quad -in\,], & n = \pm 1 \\ 0, & n \neq \pm 1 \end{cases}$$

and, therefore,

$$\mathbf{r}_{nm}^{\alpha}(\tau) = \int_{-\infty}^{\infty} \mathbf{g}_n^T(t+\tau/2)\mathbf{g}_m^*(t-\tau/2)e^{-i2\pi\alpha t}\,dt = \frac{1}{4}\delta(\tau)\begin{bmatrix} 1 & im \\ -in & nm \end{bmatrix} \quad \text{for} \quad n, m = \pm 1,$$

and

$$\mathbf{r}_{nm}^{\alpha}(\tau) \equiv 0 \quad \text{for} \quad n, m \neq \pm 1.$$

Since

$$\mathbf{R}_{\mathbf{X}}^{\alpha}(\tau) = \lim_{Z\to\infty}\frac{1}{Z}\int_{-Z/2}^{Z/2} E\left\{\begin{bmatrix} Z(t+\tau/2) \\ W(t+\tau/2) \end{bmatrix}[Z^*(t-\tau/2) \quad W^*(t-\tau/2)]\right\} e^{-i2\pi\alpha t}\,dt$$

$$= \begin{bmatrix} R_Z^{\alpha}(\tau) & R_{ZW}^{\alpha}(\tau) \\ R_{WZ}^{\alpha}(\tau) & R_W^{\alpha}(\tau) \end{bmatrix}$$

then from (12.50) we obtain (using $f_0 = 1/T_0$)

$$R_Q^{\alpha}(\tau) = \sum_{n,m=\pm 1} tr\,\{[\mathbf{R}_{\mathbf{X}}^{\alpha-(n-m)f_0}(\tau)e^{-i\pi(n+m)f_0\tau}] \otimes \mathbf{r}_{nm}^{\alpha}(-\tau)\}$$

$$= \sum_{n,m=\pm 1} tr\left\{\frac{1}{4}\begin{bmatrix} R_Z^{\alpha-(n-m)f_0}(\tau) & R_{ZW}^{\alpha-(n-m)f_0}(\tau) \\ R_{WZ}^{\alpha-(n-m)f_0}(\tau) & R_W^{\alpha-(n-m)f_0}(\tau) \end{bmatrix}\begin{bmatrix} 1 & im \\ -in & nm \end{bmatrix} e^{-i\pi(n+m)f_0\tau}\right\}$$

$$= \frac{1}{4}\sum_{n,m=\pm 1}[R_Z^{\alpha-(n-m)f_0}(\tau) - inR_{ZW}^{\alpha-(n-m)f_0}(\tau) + imR_{WZ}^{\alpha-(n-m)f_0}(\tau)$$

$$+ nmR_W^{\alpha-(n-m)f_0}(\tau)]e^{-i\pi(n+m)f_0\tau}$$

$$= \frac{1}{2}[R_Z^{\alpha}(\tau) + R_W^{\alpha}(\tau)]\cos(2\pi f_0\tau) + \frac{1}{2}[R_{WZ}^{\alpha}(\tau) - R_{ZW}^{\alpha}(\tau)]\sin(2\pi f_0\tau)$$

$$+ \sum_{n=\pm 1} \{[R_Z^{\alpha+2nf_0}(\tau) - R_W^{\alpha+2nf_0}(\tau)] + in\,[R_{WZ}^{\alpha+2nf_0}(\tau) + R_{ZW}^{\alpha+2nf_0}(\tau)]\},$$

which is the desired result (12.79).

c) To obtain the cyclic spectrum for QAM, we proceed as follows. From part *b*, we obtain

$$\mathbf{G}_n(f) = \begin{cases} \int_{-\infty}^{\infty} \mathbf{g}_n(\tau) e^{-i2\pi f\tau} d\tau = \frac{1}{2}[1 \quad -in\,], & n = \pm 1 \\ 0, & n \neq \pm 1 \end{cases}$$

and

$$\mathbf{S}_{\mathbf{X}}^{\alpha}(f) = \int_{-\infty}^{\infty} \mathbf{R}_{\mathbf{X}}^{\alpha}(\tau) e^{-i2\pi f\tau} d\tau = \begin{bmatrix} S_Z^{\alpha}(f) & S_{ZW}^{\alpha}(f) \\ S_{WZ}^{\alpha}(f) & S_W^{\alpha}(f) \end{bmatrix}$$

Therefore, we obtain from (12.53) (using $f_0 = 1/T_0$)

$$S_Q^{\alpha}(f) = \sum_{n,m=\pm 1} \mathbf{G}_n(f+\alpha/2) \mathbf{S}_{\mathbf{X}}^{\alpha-(n-m)f_0}(f - \frac{n+m}{2} f_0) \mathbf{G}_m^T(f-\alpha/2)^*$$

$$= \frac{1}{4} \sum_{n,m=\pm 1} [S_Z^{\alpha-(n-m)f_0}(f - \frac{n+m}{2} f_0) - inS_{WZ}^{\alpha-(n-m)f_0}(f - \frac{n+m}{2} f_0)$$

$$+ imS_{ZW}^{\alpha-(n-m)f_0}(f - \frac{n+m}{2} f_0) + nmS_W^{\alpha-(n-m)f_0}(f - \frac{n+m}{2} f_0)]$$

$$= \frac{1}{4} \sum_{n=\pm 1} \{[S_Z^{\alpha}(f+nf_0) + S_W^{\alpha}(f+nf_0)] + in\,[S_{WZ}^{\alpha}(f+nf_0) - S_{ZW}^{\alpha}(f+nf_0)]\}$$

$$+ \frac{1}{4} \sum_{n=\pm 1} \{[S_Z^{\alpha+2nf_0}(f) - S_W^{\alpha+2nf_0}(f)] + in\,[S_{WZ}^{\alpha+2nf_0}(f) + S_{ZW}^{\alpha+2nf_0}(f)]\},$$

which is the desired result (12.80).

d) For $Z = U$ and $W = -V$, we have

$$R_Z^{\alpha} = R_U^{\alpha}, \quad R_W^{\alpha} = R_V^{\alpha}, \quad R_{WZ}^{\alpha} = -R_{VU}^{\alpha}, \quad \text{and} \quad R_{ZW}^{\alpha} = -R_{UV}^{\alpha}.$$

Therefore, with $Q = X$ and $\alpha = 0$, (12.79) becomes

$$R_X^0(\tau) = \langle R_X \rangle(\tau) = \frac{1}{2}[\langle R_U \rangle(\tau) + \langle R_V \rangle(\tau)]\cos(2\pi f_0 \tau)$$

$$+ \frac{1}{2}[\langle R_{UV} \rangle(\tau) - \langle R_{VU} \rangle(\tau)]\sin(2\pi f_0 \tau)$$

$$+ \frac{1}{4} \sum_{n=\pm 1} \{[R_U^{2nf_0}(\tau) - R_V^{2nf_0}(\tau)] - in\,[R_{UV}^{2nf_0}(\tau) + R_{VU}^{2nf_0}(\tau)]\},$$

which is the desired result (12.81).

e) If $U(t)$ and $V(t)$ exhibit no cyclostationarity, then

$$R_U^\alpha \equiv R_V^\alpha \equiv R_{UV}^\alpha \equiv R_{VU}^\alpha \equiv 0 \quad \text{for} \quad \alpha \neq 0.$$

Therefore, for $\alpha = \pm 2f_0$ we obtain from (12.79)

$$R_X^{\pm 2f_0}(\tau) = \frac{1}{4}[\langle R_U\rangle(\tau) - \langle R_V\rangle(\tau)] \pm \frac{i}{4}[\langle R_{UV}\rangle(\tau) + \langle R_{VU}\rangle(\tau)],$$

which is the desired result (12.83a). Thus, when $U(t)$ and $V(t)$ exhibit no cyclostationarity, we have

$$R_X^\alpha(\tau) = 0, \qquad \alpha \neq 0, \pm 2f_0.$$

f) To obtain the cyclic autocorrelation and cyclic spectrum formulas for the complex envelope and the in-phase and quadrature components in Rice's representation, we proceed as follows. Using (12.301a), we obtain

$$\begin{aligned}
R_X(t+\tau/2, t-\tau/2) &= E\{X(t+\tau/2)X(t-\tau/2)\} \\
&= \frac{1}{4}E\{[\Gamma(t+\tau/2)e^{i2\pi f_0(t+\tau/2)} + \Gamma^*(t+\tau/2)e^{-i2\pi f_0(t+\tau/2)}] \\
&\quad \times [\Gamma(t-\tau/2)e^{i2\pi f_0(t-\tau/2)} + \Gamma^*(t-\tau/2)e^{-i2\pi f_0(t-\tau/2)}]\} \\
&= \frac{1}{4}R_\Gamma(t+\tau/2, t-\tau/2)e^{i2\pi f_0\tau} + \frac{1}{4}R_{\Gamma\Gamma^*}(t+\tau/2, t-\tau/2)e^{i4\pi f_0 t} \\
&\quad + \frac{1}{4}R_{\Gamma^*\Gamma}(t+\tau/2, t-\tau/2)e^{-i4\pi f_0 t} + \frac{1}{4}R_{\Gamma^*}(t+\tau/2, t-\tau/2)e^{-i2\pi f_0\tau},
\end{aligned}$$

from which it follows that

$$\begin{aligned}
R_X^\alpha(\tau) &= \lim_{Z\to\infty}\frac{1}{Z}\int_{-Z/2}^{Z/2} R_X(t+\tau/2, t-\tau/2)e^{-i2\pi\alpha t}\,dt \\
&= \frac{1}{4}R_\Gamma^\alpha(\tau)e^{i2\pi f_0\tau} + \frac{1}{4}R_{\Gamma\Gamma^*}^{\alpha-2f_0}(\tau) + \frac{1}{4}R_{\Gamma^*\Gamma}^{\alpha+2f_0}(\tau) + \frac{1}{4}R_{\Gamma^*}^\alpha(\tau)e^{-i2\pi f_0\tau}.
\end{aligned}$$

Fourier transforming $R_X^\alpha(\tau)$ yields the desired result (12.302a).

Using (12.300a), we obtain

$$\begin{aligned}
R_\Gamma(t+\tau/2, t-\tau/2) &= E\{\Gamma(t+\tau/2)\Gamma^*(t-\tau/2)\} \\
&= E\{[X(t+\tau/2) + iY(t+\tau/2)][X(t-\tau/2) - iY(t-\tau/2)]e^{-i2\pi f_0\tau}\} \\
&= [R_X(t+\tau/2, t-\tau/2) - iR_{XY}(t+\tau/2, t-\tau/2) \\
&\quad + iR_{YX}(t+\tau/2, t-\tau/2) + R_Y(t+\tau/2, t-\tau/2)]e^{-i2\pi f_0\tau}
\end{aligned}$$

and

$$R_{\Gamma\Gamma^*}(t+\tau/2,\ t-\tau/2) = E\{\Gamma(t+\tau/2)\Gamma(t-\tau/2)\}$$

$$= E\{[X(t+\tau/2) + iY(t+\tau/2)][X(t-\tau/2) + iY(t-\tau/2)]e^{-i4\pi f_0 t}\}$$

$$= [R_X(t+\tau/2,\ t-\tau/2) + iR_{XY}(t+\tau/2,\ t-\tau/2)$$

$$+ iR_{YX}(t+\tau/2,\ t-\tau/2) - R_Y(t+\tau/2,\ t-\tau/2)]e^{-i4\pi f_0 t},$$

from which it follows that

$$R_\Gamma^\alpha(\tau) = \lim_{Z\to\infty} \frac{1}{Z}\int_{-Z/2}^{Z/2} R_\Gamma(t+\tau/2,\ t-\tau/2)e^{-i2\pi\alpha t}\,dt$$

$$= [R_X^\alpha(\tau) - iR_{XY}^\alpha(\tau) + iR_{YX}^\alpha(\tau) + R_Y^\alpha(\tau)]e^{-i2\pi f_0\tau}$$

and

$$R_{\Gamma\Gamma^*}^\alpha(\tau) = R_X^{\alpha+2f_0}(\tau) + iR_{XY}^{\alpha+2f_0}(\tau) + iR_{YX}^{\alpha+2f_0}(\tau) - R_Y^{\alpha+2f_0}(\tau).$$

Fourier transforming $R_\Gamma^\alpha(\tau)$ and $R_{\Gamma\Gamma^*}^\alpha(\tau)$ yields

$$S_\Gamma^\alpha(f) = S_X^\alpha(f+f_0) - iS_{XY}^\alpha(f+f_0) + iS_{YX}^\alpha(f+f_0) + S_Y^\alpha(f+f_0) \tag{*}$$

and

$$S_{\Gamma\Gamma^*}^\alpha(f) = S_X^{\alpha+2f_0}(f) + iS_{XY}^{\alpha+2f_0}(f) + iS_{YX}^{\alpha+2f_0}(f) - S_Y^{\alpha+2f_0}(f). \tag{**}$$

Next, we reexpress (12.89) and (12.90) as

$$S_Y^\alpha(f) = -S_X^\alpha(f)[u(f+|\alpha|/2) - u(f-|\alpha|/2)] + S_X^\alpha(f)[u(f-|\alpha|/2) + u(-f-|\alpha|/2)] \tag{†}$$

$$= S_X^\alpha(f)[u(\alpha/2+|f|) - u(\alpha/2-|f|)] - S_X^\alpha(f)[u(\alpha/2-|f|) + u(-\alpha/2-|f|)] \tag{††}$$

and

$$S_{XY}^\alpha(f) = S_{YX}^\alpha(-f) = iS_X^\alpha(f)[u(f-\alpha/2) - u(-f+\alpha/2)]. \tag{†††}$$

Then we substitute the results (†) and (†††) into (*) to obtain (12.302b):

$$S_\Gamma^\alpha(f) = S_X^\alpha(f+f_0)\{1 - [u(f+f_0+|\alpha|/2) - u(f+f_0-|\alpha|/2)]$$

$$+ [u(f+f_0-|\alpha|/2) + u(-f-f_0-|\alpha|/2)]\}$$

$$+ i^2 S_X^\alpha(f+f_0)\{-[u(f+f_0-\alpha/2) - u(-f-f_0+\alpha/2)]$$

$$+ [u(-f-f_0-\alpha/2) - u(f+f_0+\alpha/2)]\} \quad (\text{using } S_X^\alpha(-f) = S_X^\alpha(f))$$

$$= 2S_X^\alpha(f+f_0)[u(f+f_0-|\alpha|/2) + u(-f-f_0-|\alpha|/2)]$$

$$+ 2S_X^\alpha(f+f_0)[u(f+f_0-|\alpha|/2) - u(-f-f_0-|\alpha|/2)]$$

$$= 4S_X^\alpha(f+f_0)u(f+f_0-|\alpha|/2),$$

which is (12.302b). Similarly, we substitute the results (††) and (†††) into (**) to obtain (12.302c):

$$S^{\alpha}_{\Gamma\Gamma^*}(f) = S_X^{\alpha+2f_0}(f)\{1 - [u(\frac{\alpha+2f_0}{2}+|f|) - u(\frac{\alpha+2f_0}{2}-|f|)]$$

$$+ [u(\frac{\alpha+2f_0}{2}-|f|) + u(-\frac{\alpha+2f_0}{2}-|f|)]\}$$

$$+ i^2 S_X^{\alpha+2f_0}(f)\{[u(f-\frac{\alpha+2f_0}{2}) - u(-f+\frac{\alpha+2f_0}{2})]$$

$$+ [u(-f-\frac{\alpha+2f_0}{2}) - u(f+\frac{\alpha+2f_0}{2})]\}$$

$$= 2S_X^{\alpha+2f_0}(f)[u(\frac{\alpha+2f_0}{2}-|f|) + u(-\frac{\alpha+2f_0}{2}-|f|)]$$

$$+ 2S_X^{\alpha+2f_0}(f)[u(\frac{\alpha+2f_0}{2}-|f|) - u(-\frac{\alpha+2f_0}{2}-|f|)]$$

$$= 4S_X^{\alpha+2f_0}(f)u(\alpha/2+f_0-|f|),$$

which is (12.302c).

Since (using (12.301b)-(12.301c))

$$R_U^{\alpha}(\tau) = \langle E\{U(t+\tau/2)U^*(t-\tau/2)\}e^{-i2\pi\alpha t}\rangle$$

$$= \frac{1}{4}R^{\alpha}_{\Gamma}(\tau) + \frac{1}{4}R^{\alpha}_{\Gamma\Gamma^*}(\tau) + \frac{1}{4}R^{\alpha}_{\Gamma^*\Gamma}(\tau) + \frac{1}{4}R^{\alpha}_{\Gamma^*}(\tau),$$

$$R_V^{\alpha}(\tau) = \langle E\{V(t+\tau/2)V^*(t-\tau/2)\}e^{-i2\pi\alpha t}\rangle$$

$$= \frac{1}{4}R^{\alpha}_{\Gamma}(\tau) - \frac{1}{4}R^{\alpha}_{\Gamma\Gamma^*}(\tau) - \frac{1}{4}R^{\alpha}_{\Gamma^*\Gamma}(\tau) + \frac{1}{4}R^{\alpha}_{\Gamma^*}(\tau),$$

and

$$R_{UV}^{\alpha}(\tau) = \langle E\{U(t+\tau/2)V^*(t-\tau/2)\}e^{-i2\pi\alpha t}\rangle$$

$$= \frac{i}{4}R^{\alpha}_{\Gamma}(\tau) - \frac{i}{4}R^{\alpha}_{\Gamma\Gamma^*}(\tau) + \frac{i}{4}R^{\alpha}_{\Gamma^*\Gamma}(\tau) - \frac{i}{4}R^{\alpha}_{\Gamma^*}(\tau),$$

then we have the corresponding Fourier transforms

$$S_U^{\alpha}(f) = \frac{1}{4}[S^{\alpha}_{\Gamma}(f) + S^{\alpha}_{\Gamma\Gamma^*}(f) + S^{\alpha}_{\Gamma^*\Gamma}(f) + S^{\alpha}_{\Gamma^*}(f)],$$

$$S_V^{\alpha}(f) = \frac{1}{4}[S^{\alpha}_{\Gamma}(f) - S^{\alpha}_{\Gamma\Gamma^*}(f) - S^{\alpha}_{\Gamma^*\Gamma}(f) + S^{\alpha}_{\Gamma^*}(f)],$$

and

$$S_{UV}^{\alpha}(f) = \frac{i}{4}[S^{\alpha}_{\Gamma}(f) - S^{\alpha}_{\Gamma\Gamma^*}(f) + S^{\alpha}_{\Gamma^*\Gamma}(f) - S^{\alpha}_{\Gamma^*}(f)],$$

which are the desired results (12.302d)-(12.302f).

Finally, substituting (12.302b)-(12.302c) into (12.302d)-(12.302f) yields (using $S_{\Gamma^*}^{\alpha}(f) = [S_{\Gamma}^{-\alpha}(-f)]^*$ and $S_{\Gamma^*\Gamma}^{\alpha}(f) = [S_{\Gamma\Gamma^*}^{-\alpha}(-f)]^*$)

$$S_U^\alpha(f) = \frac{1}{4}[4S_X^\alpha(f+f_0)u(f+f_0-|\alpha|/2) + 4S_X^{\alpha+2f_0}(f)u(\alpha/2+f_0-|f|)$$

$$+ 4S_X^{-\alpha+2f_0}(-f)^*u(-\alpha/2+f_0-|f|) + 4S_X^{-\alpha}(-f+f_0)^*u(-f+f_0-|\alpha|/2)]$$

$$= S_X^\alpha(f+f_0)u(f+f_0-|\alpha|/2) + S_X^\alpha(f-f_0)u(-f+f_0-|\alpha|/2)$$

$$+ S_X^{\alpha+2f_0}(f)u(\alpha/2+f_0-|f|) + S_X^{\alpha-2f_0}(f)u(-\alpha/2+f_0-|f|),$$

$$S_V^\alpha(f) = \frac{1}{4}[4S_X^\alpha(f+f_0)u(f+f_0-|\alpha|/2) - 4S_X^{\alpha+2f_0}(f)u(\alpha/2+f_0-|f|)$$

$$- 4S_X^{-\alpha+2f_0}(-f)^*u(-\alpha/2+f_0-|f|) + 4S_X^{-\alpha}(-f+f_0)^*u(-f+f_0-|\alpha|/2)]$$

$$= S_X^\alpha(f+f_0)u(f+f_0-|\alpha|/2) + S_X^\alpha(f-f_0)u(-f+f_0-|\alpha|/2)$$

$$- S_X^{\alpha+2f_0}(f)u(\alpha/2+f_0-|f|) - S_X^{\alpha-2f_0}(f)u(-\alpha/2+f_0-|f|),$$

and

$$S_{UV}^\alpha(f) = \frac{i}{4}[4S_X^\alpha(f+f_0)u(f+f_0-|\alpha|/2) - 4S_X^{\alpha+2f_0}(f)u(\alpha/2+f_0-|f|)$$

$$+ 4S_X^{-\alpha+2f_0}(-f)^*u(-\alpha/2+f_0-|f|) - 4S_X^{-\alpha}(-f+f_0)^*u(-f+f_0-|\alpha|/2)]$$

$$= i\,[S_X^\alpha(f+f_0)u(f+f_0-|\alpha|/2) - S_X^\alpha(f-f_0)u(-f+f_0-|\alpha|/2)$$

$$- S_X^{\alpha+2f_0}(f)u(\alpha/2+f_0-|f|) + S_X^{\alpha-2f_0}(f)u(-\alpha/2+f_0-|f|)],$$

which are the desired results (12.91)-(12.92).

12.9 *a*) The synchronized averaging identity (12.42) can be expressed more generally as

$$\lim_{N\to\infty}\frac{1}{2N+1}\sum_{n=-N}^{N} f(nT) = \sum_{m=-\infty}^{\infty}\lim_{Z\to\infty}\frac{1}{Z}\int_{-Z/2}^{Z/2} f(t)e^{-i2\pi mt/T}\,dt$$

for any function $f(t)$ for which these averages exist. By using the identity $\tilde{R}_X(nT+kT,\, nT) = R_X(nT+kT,\, nT)$ in (12.94) and then considering

$$f(nT) = R_X(nT+kT,\, nT)e^{-i2\pi\alpha(nT+kT/2)},$$

we obtain

$$\tilde{R}_X^\alpha(kT) = \sum_{m=-\infty}^{\infty}\lim_{Z\to\infty}\frac{1}{Z}\int_{-Z/2}^{Z/2} R_X(t+kT,\, t)e^{-i2\pi\alpha(t+kT/2)}e^{-i2\pi mt/T}\,dt$$

$$= \sum_{m=-\infty}^{\infty} \lim_{Z\to\infty} \frac{1}{Z} \int_{-Z/2}^{Z/2} R_X(t+kT,\, t) e^{-i2\pi(\alpha+m/T)t}\, dt\ e^{-i\pi\alpha kT}.$$

Letting $s = t + kT/2$, this equation becomes

$$\tilde{R}_X^\alpha(kT) = \sum_{m=-\infty}^{\infty} \lim_{Z\to\infty} \frac{1}{Z} \int_{-Z/2+kT/2}^{Z/2+kT/2} R_X(s+kT/2,\, s-kT/2) e^{-i2\pi(\alpha+m/T)s}\, ds e^{i\pi mk}$$

$$= \sum_{m=-\infty}^{\infty} R_X^{\alpha+m/T}(kT) e^{i\pi mk},$$

which is the desired cyclic autocorrelation relation for periodic time sampling.

b) Substituting (12.96) into the definition (12.95) yields

$$\tilde{S}_X^\alpha(f) = \sum_{k=-\infty}^{\infty} \tilde{R}_X^\alpha(kT) e^{-i2\pi kTf} = \sum_{m=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} R_X^{\alpha+m/T}(kT) e^{i\pi mk} e^{-i2\pi kTf}$$

$$= \int_{-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} S_X^{\alpha+m/T}(\nu) e^{i2\pi\nu kT} e^{i\pi mk} e^{-i2\pi kTf}\, d\nu.$$

Using the identity

$$\sum_{k=-\infty}^{\infty} e^{i2\pi\mu kT} \equiv \frac{1}{T} \sum_{n=-\infty}^{\infty} \delta(\mu + \frac{n}{T})$$

then yields

$$\tilde{S}_X^\alpha(f) = \int_{-\infty}^{\infty} \frac{1}{T} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} S_X^{\alpha+m/T}(\nu) \delta(\nu - f + \frac{m}{2T} + \frac{n}{T}) d\nu$$

$$= \frac{1}{T} \sum_{n,m=-\infty}^{\infty} S_X^{\alpha+m/T}(f - \frac{m}{2T} - \frac{n}{T}),$$

which is the desired cyclic spectrum aliasing formula (12.97).

From (12.97) we observe that $\tilde{S}_X^\alpha(f)$ has the following periodicity properties:

$$\tilde{S}_X^\alpha(f + \frac{1}{T}) = \frac{1}{T} \sum_{n,m=-\infty}^{\infty} S_X^{\alpha+m/T}(f - \frac{m}{2T} - \frac{n-1}{T})$$

$$= \frac{1}{T} \sum_{n',m=-\infty}^{\infty} S_X^{\alpha+m/T}(f - \frac{m}{2T} - \frac{n'}{T}) = \tilde{S}_X^\alpha(f),$$

$$\tilde{S}_X^{\alpha+2/T}(f) = \frac{1}{T} \sum_{n,m=-\infty}^{\infty} S_X^{\alpha+2/T+m/T}(f - \frac{m}{2T} - \frac{n}{T})$$

$$= \frac{1}{T} \sum_{n,m'=-\infty}^{\infty} S_X^{\alpha+m'/T}(f - \frac{m'-2}{2T} - \frac{n}{T})$$

$$= \frac{1}{T} \sum_{n,m'=-\infty}^{\infty} S_X^{\alpha+m'/T}(f - \frac{m'}{2T} - \frac{n-1}{T})$$

$$= \frac{1}{T} \sum_{n',m'=-\infty}^{\infty} S_X^{\alpha+m'/T}(f - \frac{m'}{2T} - \frac{n'}{T}) = \tilde{S}_X^{\alpha}(f),$$

and

$$\tilde{S}_X^{\alpha+1/T}(f - \frac{1}{2T}) = \frac{1}{T} \sum_{n,m=-\infty}^{\infty} S_X^{\alpha+1/T+m/T}(f - \frac{1}{2T} - \frac{m}{2T} - \frac{n}{T})$$

$$= \frac{1}{T} \sum_{n,m'=-\infty}^{\infty} S_X^{\alpha+m'/T}(f - \frac{m'}{2T} - \frac{n}{T})$$

$$= \frac{1}{T} \sum_{n',m'=-\infty}^{\infty} S_X^{\alpha+m'/T}(f - \frac{m'}{2T} - \frac{n'}{T}) = \tilde{S}_X^{\alpha}(f).$$

12.10 *a*) We first express the AM sine wave (12.103) in the form of (12.44):

$$X(t) = \int_{-\infty}^{\infty} A(u) cos(2\pi f_0 u + \phi_0)\delta(t-u)du = \int_{-\infty}^{\infty} A(u)h(t, u)du,$$

where

$$h(t, u) = cos(2\pi f_0 u + \phi_0)\delta(t-u).$$

Then, from (12.45) and (12.46), we obtain

$$h(t+\tau, t) = \cos(2\pi f_0 t + \phi_0)\delta(\tau) = \frac{1}{2}[e^{i(2\pi f_0 t + \phi_0)} + e^{-i(2\pi f_0 t + \phi_0)}]\delta(\tau)$$

and

$$g_n(\tau) = \begin{cases} \frac{1}{2}e^{i\phi_0}\delta(\tau), & n = 1 \\ \frac{1}{2}e^{-i\phi_0}\delta(\tau), & n = -1. \end{cases}$$

From (12.52), we obtain

$$r_{nm}^{\alpha}(\tau) = \int_{-\infty}^{\infty} g_n(t+\tau/2)g_m^*(t-\tau/2)e^{-i2\pi\alpha t}dt = \frac{1}{4}e^{i(n-m)\phi_0}\delta(\tau), \quad n, m = \pm 1.$$

Finally, substituting this result into (12.50) with $T = T_0 = 1/f_0$ yields the cyclic autocorrelation for the AM sine wave $X(t)$:

$$R_X^{\alpha}(\tau) = \sum_{n,m=\pm 1} [R_A^{\alpha-(n-m)/T_0}(\tau)e^{-i\pi(n+m)\tau/T_0}] \otimes [\frac{1}{4}e^{i(n-m)\phi_0}\delta(-\tau)]$$

$$= \frac{1}{4} \sum_{n,m=\pm 1} R_A^{\alpha-(n-m)/T_0}(\tau)e^{-i\pi(n+m)\tau/T_0}e^{i(n-m)\phi_0}$$

$$= \frac{1}{4}[R_A^{\alpha}(\tau)e^{-i2\pi f_0\tau} + R_A^{\alpha}(\tau)e^{i2\pi f_0\tau} + R_A^{\alpha-2f_0}(\tau)e^{i2\phi_0} + R_A^{\alpha+2f_0}(\tau)e^{-i2\phi_0}]$$

$$= \frac{1}{2} R_A^{\alpha}(\tau) \cos(2\pi f_0 \tau) + \frac{1}{4} R_A^{\alpha - 2f_0}(\tau) e^{i2\phi_0} + \frac{1}{4} R_A^{\alpha + 2f_0}(\tau) e^{-i2\phi_0}.$$

b) Fourier transforming the cyclic autocorrelation (12.104) yields the cyclic spectrum (12.105) for the AM sine wave:

$$S_X^{\alpha}(f) = \int_{-\infty}^{\infty} R_X^{\alpha}(\tau) e^{-i2\pi f \tau} d\tau$$

$$= \frac{1}{4} [S_A^{\alpha}(f + f_0) + S_A^{\alpha}(f - f_0) + S_A^{\alpha + 2f_0}(f) e^{-i2\phi_0} + S_A^{\alpha - 2f_0}(f) e^{i2\phi_0}],$$

where

$$S_A^{\alpha}(f) = \int_{-\infty}^{\infty} R_A^{\alpha}(\tau) e^{-i2\pi f \tau} d\tau.$$

12.11 *a*) For the QAM signal, making the replacements $Q = X$, $Z = C$, and $W = -S$ in (12.80) yields the cyclic spectrum

$$S_X^{\alpha}(f) = \frac{1}{4} \sum_{n=\pm 1} \{S_S^{\alpha}(f + nf_0) + S_C^{\alpha}(f + nf_0) + in [S_{CS}^{\alpha}(f + nf_0) - S_{SC}^{\alpha}(f + nf_0)]\}$$

$$+ \frac{1}{4} \sum_{n=\pm 1} \{S_C^{\alpha + 2nf_0}(f) - S_S^{\alpha + 2nf_0}(f) - in [S_{SC}^{\alpha + 2nf_0}(f) + S_{CS}^{\alpha + 2nf_0}(f)]\}.$$

For $C(t)$ and $S(t)$ jointly stationary, $S_X^{\alpha}(f)$ at $\alpha = 0$ becomes

$$S_X^0(f) = \langle S_X \rangle(f) = \frac{1}{4} [S_C(f + f_0) + S_C(f - f_0) + S_S(f + f_0) + S_S(f - f_0)]$$

$$+ \frac{i}{4} [S_{CS}(f + f_0) - S_{SC}(f + f_0)] - \frac{i}{4} [S_{CS}(f - f_0) - S_{SC}(f - f_0)]$$

$$= \frac{1}{4} [S_C(f + f_0) + S_C(f - f_0) + S_S(f + f_0) + S_S(f - f_0)]$$

$$- \frac{1}{2} [S_{CS}(f + f_0)_i - S_{CS}(f - f_0)_i] \quad \text{(using } S_{SC}(f) = S_{CS}^*(f)\text{)},$$

and $S_X^{\alpha}(f)$ at $\alpha = \pm 2f_0$ becomes

$$S_X^{\alpha}(f) = \frac{1}{4} [S_C(f) - S_S(f)] \pm \frac{i}{4} [S_{SC}(f) + S_{CS}(f)]$$

$$= \frac{1}{4} [S_C(f) - S_S(f)] \pm \frac{i}{2} S_{CS}(f)_r, \qquad \alpha = \pm 2f_0$$

and

$$S_X^{\alpha}(f) = 0 \quad \text{for} \quad \alpha \neq 0 \quad \text{and} \quad \alpha \neq \pm 2f_0.$$

This verifies the cyclic spectrum formulas (12.114) and (12.115) for the QAM signal.

b) From (12.17), we have

$$|\rho_X^\alpha(f)|^2 = \frac{|S_X^\alpha(f)|^2}{<S_X>(f+\alpha/2)<S_X>(f-\alpha/2)}$$

$$= \frac{\frac{1}{16}[S_C(f) - S_S(f)]^2 + \frac{1}{4}[S_{CS}(f)_r]^2}{\{\frac{1}{4}[S_C(f)+S_S(f)] + \frac{1}{2}S_{CS}(f)_i\}\{\frac{1}{4}[S_C(f)+S_S(f)] - \frac{1}{2}S_{CS}(f)_i\}}$$

$$= \frac{[S_C(f) - S_S(f)]^2 + 4[S_{CS}(f)_r]^2}{[S_C(f)+S_S(f)]^2 - 4[S_{CS}(f)_i]^2}, \qquad \alpha = \pm 2f_0,$$

which is the desired formula for the autocoherence magnitude for QAM.

c) Equation (12.17) can be reexpressed as

$$|\rho_X^\alpha(f)|^2 = 1 + \frac{4[S_{CS}(f)_i]^2 + 4[S_{CS}(f)_r]^2 + [S_C(f)-S_S(f)]^2 - [S_C(f)+S_S(f)]^2}{[S_C(f)+S_S(f)]^2 - 4[S_{CS}(f)_i]^2}$$

$$= 1 + \frac{4|S_{CS}(f)|^2 - 4S_C(f)S_S(f)}{[S_C(f)+S_S(f)]^2 - 4[S_{CS}(f)_i]^2}$$

$$= 1 - \frac{4(1 - |\rho_{CS}(f)|^2)S_C(f)S_S(f)}{[S_C(f)+S_S(f)]^2 - 4[S_{CS}(f)_i]^2}, \qquad \alpha = \pm 2f_0,$$

which is the desired result (12.118).

d) Since

$$<S_X>(f-f_0) = \frac{1}{4}[S_C(f) + S_S(f)] - \frac{1}{2}S_{CS}(f)_i$$

$$= <S_X>(f-f_0)_e - <S_X>(f-f_0)_o, \qquad |f| \le f_0,$$

where

$$<S_X>(f-f_0)_e = \frac{1}{4}[S_C(f) + S_S(f)] = <S_X>(-f-f_0)_e$$

$$<S_X>(f-f_0)_o = \frac{1}{2}S_{CS}(f)_i = -<S_X>(-f-f_0)_o$$

(using $S_{CS}(-f) = S_{CS}^*(f)$), then the denominator in (12.118) can be expressed as (12.119).

12.12 To derive the stationarity condition (12.126), we proceed as follows. From (12.113c), we have

$$R_C(\tau) = E\{C(t+\tau/2)C(t-\tau/2)\}$$
$$= \frac{1}{2}E\{A(t+\tau/2)A(t-\tau/2)\big(\cos[\Phi(t+\tau/2) + \Phi(t-\tau/2)]$$
$$+ \cos[\Phi(t+\tau/2) - \Phi(t-\tau/2)]\big)\}$$

and

$$R_S(\tau) = E\{S(t+\tau/2)S(t-\tau/2)\}$$
$$= \frac{1}{2}E\{A(t+\tau/2)A(t-\tau/2)\big(\cos[\Phi(t+\tau/2) - \Phi(t-\tau/2)]$$
$$- \cos[\Phi(t+\tau/2) + \Phi(t-\tau/2)]\big)\}.$$

Therefore, in order for $S_C(f) = S_S(f)$ or, equivalently, for $R_C(\tau) = R_S(\tau)$ to hold, we require

$$E\{A(t+\tau/2)A(t-\tau/2)\cos[\Phi(t+\tau/2) + \Phi(t-\tau/2)]\} = 0. \qquad (*)$$

Likewise, since

$$R_{CS}(\tau) = E\{C(t+\tau/2)S(t-\tau/2)\}$$
$$= \frac{1}{2}E\{A(t+\tau/2)A(t-\tau/2)\big(\sin[\Phi(t+\tau/2) + \Phi(t-\tau/2)]$$
$$- \sin[\Phi(t+\tau/2) - \Phi(t-\tau/2)]\big)\},$$

then in order for $S_{CS}(f)_r = 0$ or, equivalently, $R_{CS}(\tau)_e = 0$, we require

$$E\{A(t+\tau/2)A(t-\tau/2)\sin[\Phi(t+\tau/2) + \Phi(t-\tau/2)]\} = 0. \qquad (**)$$

The conditions (*) and (**) together are equivalent to (12.126).

12.13 Directly applying the definition of cyclic autocorrelation to the PM-FM signal $X(t)$ in (12.129) yields

$$R_X^\alpha(\tau) = \langle E\{X(t+\tau/2)X(t-\tau/2)\}e^{-i2\pi\alpha t}\rangle$$
$$= \frac{1}{2}\langle E\{\cos[(2\pi f_0\tau + \Phi(t+\tau/2) - \Phi(t-\tau/2)]\}e^{-i2\pi\alpha t}\rangle$$
$$+ \frac{1}{2}\langle E\{\cos[4\pi f_0 t + \Phi(t+\tau/2) + \Phi(t-\tau/2)]\}e^{-i2\pi\alpha t}\rangle$$
$$= 0 + \frac{1}{4}\langle \exp\{i4\pi f_0 t\}E\{\exp\{i\Phi(t+\tau/2) + i\Phi(t-\tau/2)\}\}e^{-i2\pi\alpha t}\rangle$$
$$+ \frac{1}{4}\langle \exp\{-i4\pi f_0 t\}E\{\exp\{-i\Phi(t+\tau/2) - i\Phi(t-\tau/2)\}\}e^{-i2\pi\alpha t}\rangle$$

$$
= \begin{cases} \frac{1}{4} <E\{\exp\{i\Phi(t+\tau/2)+i\Phi(t-\tau/2)\}\}>, & \alpha = 2f_0 \\ \frac{1}{4} <E\{\exp\{-i\Phi(t+\tau/2)-i\Phi(t-\tau/2)\}\}>, & \alpha = -2f_0 \\ 0, & \alpha \neq 0, \pm 2f_0 \end{cases}
$$

which is the desired result (12.130).

12.14 *a*) Substituting $h(t, u)$ from (12.135) and $X(u) = A(u)$ into (12.44) yields

$$
X(t) = \int_{-\infty}^{\infty} h(t, u) A(u) du = \int_{-\infty}^{\infty} \sum_{m=-\infty}^{\infty} q(t - t_0 - mT)\delta(u - mT) A(u) du
$$

$$
= \sum_{m=-\infty}^{\infty} A(mT) q(t - t_0 - mT),
$$

which is the PAM signal (12.134).

b) Substituting (12.135) into (12.46) yields

$$
g_n(\tau) = \frac{1}{T} \int_{-T/2}^{T/2} h(t+\tau, t) e^{-i2\pi nt/T} dt
$$

$$
= \frac{1}{T} \int_{-T/2}^{T/2} \sum_{m=-\infty}^{\infty} q(t+\tau-t_0-mT)\delta(t-mT) e^{-i2\pi nt/T} dt
$$

$$
= \frac{1}{T} \sum_{m=-\infty}^{\infty} \int_{mT-T/2}^{mT+T/2} q(s+\tau-t_0)\delta(s) e^{-i2\pi ns/T} ds
$$

$$
= \frac{1}{T} \int_{-\infty}^{\infty} q(s+\tau-t_0)\delta(s) e^{-i2\pi ns/T} ds = \frac{1}{T} q(\tau - t_0),
$$

which is the desired result (12.136).

c) From (12.52), we have

$$
r_{nm}^{\alpha}(\tau) = \int_{-\infty}^{\infty} g_n(t+\tau/2) g_m^*(t-\tau/2) e^{-i2\pi\alpha t} dt
$$

$$
= \frac{1}{T^2} \int_{-\infty}^{\infty} q(t+\tau/2-t_0) q(t-\tau/2-t_0) e^{-i2\pi\alpha t} dt = \frac{1}{T^2} r_q^{\alpha}(\tau) e^{-i2\pi\alpha t_0}.
$$

Therefore, by using (12.50) and (12.137), we obtain

$$
R_X^{\alpha}(\tau) = \frac{1}{T^2} [\sum_{n,m=-\infty}^{\infty} R_A^{\alpha-(n-m)/T}(\tau) e^{-i\pi(n+m)\tau/T}] \otimes r_q^{\alpha}(-\tau) e^{-i2\pi\alpha t_0}
$$

$$
= \frac{1}{T^2} [\sum_{n,k=-\infty}^{\infty} R_A^{\alpha-k/T}(\tau) e^{-i\pi(2n-k)\tau/T}] \otimes r_q^{\alpha}(\tau) e^{-i2\pi\alpha t_0}
$$

$$= \frac{1}{T^2}[T \sum_{j,k=-\infty}^{\infty} R_A^{\alpha-k/T}(\tau)\delta(\tau+jT)e^{i\pi k\tau/T}] \otimes r_q^{\alpha}(\tau)e^{-i2\pi\alpha t_0}$$

$$= \frac{1}{T}\sum_{j,k=-\infty}^{\infty} R_A^{\alpha-k/T}(jT)r_q^{\alpha}(\tau+jT)e^{-i2\pi(\alpha t_0+jk/2)},$$

which is the desired formula (12.138) for the cyclic autocorrelation of PAM.

d) By using (12.96), (12.138) can be reexpressed as

$$R_X^{\alpha}(\tau) = \frac{1}{T}\sum_{j=-\infty}^{\infty}[\sum_{k=-\infty}^{\infty} R_A^{\alpha+k/T}(jT)e^{i\pi jk}]r_q^{\alpha}(\tau+jT)e^{-i2\pi\alpha t_0}$$

$$= \frac{1}{T}\sum_{j=-\infty}^{\infty}\tilde{R}_A^{\alpha}(jT)r_q^{\alpha}(\tau+jT)e^{-i2\pi\alpha t_0},$$

which is the desired result (12.139).

e) Directly applying to (12.138) the generalized Poisson sum formula with

$$k(\tau) = R_X^{\alpha}(\tau), \quad h(jT) = \frac{1}{T}\sum_{k=-\infty}^{\infty} R_A^{\alpha-k/T}(jT)e^{-i\pi jk}$$

and

$$g(\tau+jT) = r_q^{\alpha}(\tau+jT)e^{-i2\pi\alpha t_0},$$

which have the corresponding Fourier transforms

$$K(f) = S_X^{\alpha}(f), \quad H(f) = \frac{1}{T}\sum_{k=-\infty}^{\infty} S_A^{\alpha-k/T}(f+\frac{k}{2T}),$$

and

$$G(f) = Q(f+\alpha/2)Q^*(f-\alpha/2)e^{-i2\pi\alpha t_0},$$

yields

$$S_X^{\alpha}(f) = K(f) = \frac{1}{T}G(f)\sum_{n=-\infty}^{\infty} H(f+\frac{n}{T})$$

$$= \frac{1}{T^2}Q(f+\alpha/2)Q^*(f-\alpha/2)e^{-i2\pi\alpha t_0}\sum_{n,k=-\infty}^{\infty} S_A^{\alpha-k/T}(f+\frac{n}{T}+\frac{k}{2T}),$$

which is the desired formula for the cyclic spectrum for PAM.

f) Substituting (12.97) into (12.142) immediately yields (12.143).

12.15 *a*) To obtain the formula for the cyclic autocorrelation for a digital modulated pulse train, we proceed as follows. We first reexpress (12.146) in the form of (12.44)

$$X(t) = \int_{-\infty}^{\infty} \mathbf{h}(t,u)\mathbf{Y}(u)du,$$

for which

$$\mathbf{h}(t, u) = \sum_{n=-\infty}^{\infty} \mathbf{q}^T(t-u)\delta(u-nT)$$

and

$$\mathbf{Y}(nT) = \boldsymbol{\delta}(n),$$

Using the results from exercise 12.14*a* and 12.14*b*, we have (in vector form)

$$\mathbf{g}_n(\tau) = \frac{1}{T}\int_{-T/2}^{T/2} \mathbf{h}(t+\tau,\, t)e^{-i2\pi nt/T}\, dt = \frac{1}{T}\mathbf{q}(\tau)$$

and

$$\mathbf{r}_{nm}^{\alpha}(\tau) = \int_{-\infty}^{\infty} \mathbf{g}_n^T(t+\tau/2)\mathbf{g}_m^*(t-\tau/2)e^{-i2\pi\alpha t}\, dt$$

$$= \frac{1}{T^2}\int_{-\infty}^{\infty} \mathbf{q}^T(t+\tau/2)\mathbf{q}^*(t-\tau/2)e^{-i2\pi\alpha t}\, dt = \frac{1}{T^2}\mathbf{r}_{\mathbf{q}}^{\alpha}(\tau).$$

Also, we have

$$\sum_{l=-\infty}^{\infty} \mathbf{R}_{\mathbf{Y}}^{\alpha-l/T}(nT)e^{i\pi ln} = \tilde{\mathbf{R}}_{\boldsymbol{\delta}}^{\alpha}(n).$$

Thus, from (12.50), we obtain

$$R_X^{\alpha}(\tau) = \sum_{n,m=-\infty}^{\infty} tr\,\{[\mathbf{R}_{\mathbf{Y}}^{\alpha-(n-m)/T}(\tau)e^{-i\pi(n+m)\tau/T}] \otimes \mathbf{r}_{nm}^{\alpha}(-\tau)\}$$

$$= \frac{1}{T^2}\sum_{n,m=-\infty}^{\infty} tr\,\{[\mathbf{R}_{\mathbf{Y}}^{\alpha-(n-m)/T}(\tau)e^{-i\pi(n+m)\tau/T}] \otimes \mathbf{r}_{\mathbf{q}}^{\alpha}(-\tau)\}$$

$$= \frac{1}{T^2}\sum_{l=-\infty}^{\infty} tr\,\{[\mathbf{R}_{\mathbf{Y}}^{\alpha-l/T}(\tau)e^{-i\pi l\tau/T} \sum_{m=-\infty}^{\infty} e^{-i2\pi m\tau/T}] \otimes \mathbf{r}_{\mathbf{q}}^{\alpha}(-\tau)\}$$

$$= \frac{1}{T^2}\sum_{l=-\infty}^{\infty} tr\,\{[\mathbf{R}_{\mathbf{Y}}^{\alpha-l/T}(\tau)e^{-i\pi l\tau/T}\; T\sum_{n=-\infty}^{\infty} \delta(\tau-nT)] \otimes \mathbf{r}_{\mathbf{q}}^{\alpha}(-\tau)\}$$

$$= \frac{1}{T} tr\,\{[\sum_{n=-\infty}^{\infty} \delta(\tau-nT) \sum_{l=-\infty}^{\infty} \mathbf{R}_{\mathbf{Y}}^{\alpha-l/T}(nT)e^{-i\pi ln}] \otimes \mathbf{r}_{\mathbf{q}}^{\alpha}(-\tau)\}$$

$$= \frac{1}{T}\sum_{n=-\infty}^{\infty} tr\,\{[\delta(\tau-nT)\tilde{\mathbf{R}}_{\boldsymbol{\delta}}^{\alpha}(n)] \otimes \mathbf{r}_{\mathbf{q}}^{\alpha}(-\tau)\} \quad \text{(using (12.96))}$$

$$= \frac{1}{T}\sum_{n=-\infty}^{\infty} tr\,\{\tilde{\mathbf{R}}_{\boldsymbol{\delta}}^{\alpha}(n)\mathbf{r}_{\mathbf{q}}^{\alpha}(nT-\tau)\}$$

$$= \frac{1}{T}\sum_{j=-\infty}^{\infty} tr\,\{\mathbf{r}_{\mathbf{q}}^{\alpha}(\tau+jT)\tilde{\mathbf{R}}_{\boldsymbol{\delta}}^{\alpha}(j)\},$$

which is the desired result (12.147) for the cyclic autocorrelation for a digital modulated pulse train.

b) If $\{\delta(n)\}$ exhibits no cyclostationarity, then we have from (12.148)

$$\tilde{\mathbf{R}}_\delta^0(j) = <\tilde{\mathbf{R}}_\delta>(j) = \tilde{\mathbf{R}}_\delta^\alpha(j)e^{-i\pi\alpha jT}, \quad \alpha = k/T.$$

Therefore, (12.147) reduces to (12.150).

c) Fourier transforming (12.150) yields

$$\begin{aligned} S_X^\alpha(f) &= \frac{1}{T}\sum_{j=-\infty}^{\infty} tr\,\{\mathbf{Q}(f+\alpha/2)\mathbf{Q}^T(f-\alpha/2)^* e^{i2\pi fjT}<\tilde{\mathbf{R}}_\delta>(j)\}e^{i\pi\alpha jT} \\ &= \frac{1}{T}tr\,\{\mathbf{Q}(f+\alpha/2)\mathbf{Q}^T(f-\alpha/2)^* \sum_{j=-\infty}^{\infty} <\tilde{\mathbf{R}}_\delta>(j)e^{i2\pi(f+\alpha/2)jT}\} \\ &= \frac{1}{T}tr\,\{\mathbf{Q}(f+\alpha/2)\mathbf{Q}^T(f-\alpha/2)^* <\tilde{\mathbf{S}}_\delta>(f+\alpha/2)^T\} \\ &= \frac{1}{T}\mathbf{Q}^T(f+\alpha/2)<\tilde{\mathbf{S}}_\delta>(f+\alpha/2)\mathbf{Q}^*(f-\alpha/2), \quad \alpha = k/T, \end{aligned}$$

which is the desired formula (12.153) for the cyclic spectrum.

12.16 For the FSK signal, applying the definition of autocorrelation and the conditional expectation property (2.45) to $X(t)$ in (12.159) yields

$$R_X(t+\tau/2,\, t-\tau/2) \;=\; \sum_{n,m=-\infty}^{\infty} E\{A_n(t+\tau/2)A_m(t-\tau/2)\}q(t-nT_c+\tau/2)q(t-mT_c-\tau/2),$$

for which

$$\begin{aligned} E\{A_n(t+\tau/2)A_m(t-\tau/2)\} &= \frac{1}{2}E\{E\{\cos[2\pi(F_n+F_m)t+\pi(F_n-F_m)\tau+\Theta_n+\Theta_m] \\ &\quad + \cos[2\pi(F_n-F_m)t+\pi(F_n+F_m)\tau+\Theta_n-\Theta_m]\}\,|F_n,F_m\} \\ &= \frac{1}{2}E\{\cos(2\pi F_n\tau)\}\delta_{n-m}. \end{aligned}$$

Therefore, we have the desired result (12.160), in which $E\{\cos(2\pi F_n\tau)\}$ is independent of n.

12.17 *a*) The staggered QPSK signal has the form

$$X(t) = C(t)\cos(2\pi f_0 t+\phi_0) - S(t)\sin(2\pi f_0 t+\phi_0),$$

for which

$$C(t) = \sum_{n=-\infty}^{\infty} C(nT_c)q(t-t_0-nT_c-T_c/2)$$

$$S(t) = \sum_{n=-\infty}^{\infty} S(nT_c)q(t - t_0 - nT_c),$$

where $\{C(nT_c)\}$ and $\{S(nT_c)\}$ are two independent zero-mean binary sequences with values ± 1. Since $C(t)$ and $S(t)$ are each PAM signals, then from (12.143) we have

$$S_C^\alpha(f) = \frac{1}{T_c} Q(f + \alpha/2) Q^*(f - \alpha/2) \tilde{S}_C^\alpha(f) e^{-i\pi\alpha T_c} e^{-i2\pi\alpha t_0}$$

and

$$S_S^\alpha(f) = \frac{1}{T_c} Q(f + \alpha/2) Q^*(f - \alpha/2) \tilde{S}_S^\alpha(f) e^{-i2\pi\alpha t_0},$$

where $\tilde{S}_C^\alpha(f) \equiv \tilde{S}_S^\alpha(f)$; since $C(t)$ and $S(t)$ are independent and have zero mean values, then

$$S_{SC}^\alpha(f) = S_{CS}^\alpha(f) \equiv 0.$$

Substituting these results into (12.80) with $Z = C$ and $W = -S$ yields the cyclic spectrum for this staggered QPSK signal:

$$\begin{aligned}
S_X^\alpha(f) &= \frac{1}{4}[S_C^\alpha(f + f_0) + S_C^\alpha(f - f_0) + S_S^\alpha(f + f_0) + S_S^\alpha(f - f_0)] \\
&\quad + \frac{1}{4}[S_C^{\alpha+2f_0}(f) - S_S^{\alpha+2f_0}(f)]e^{-i2\phi_0} + \frac{1}{4}[S_C^{\alpha-2f_0}(f) - S_S^{\alpha-2f_0}(f)]e^{i2\phi_0} \\
&= \frac{1}{4T_c} Q(f + f_0 + \alpha/2) Q^*(f + f_0 - \alpha/2) \tilde{S}_C^\alpha(f + f_0)[1 + e^{-i\pi\alpha T_c}]e^{-i2\pi\alpha t_0} \\
&\quad + \frac{1}{4T_c} Q(f - f_0 + \alpha/2) Q^*(f - f_0 - \alpha/2) \tilde{S}_C^\alpha(f - f_0)[1 + e^{-i\pi\alpha T_c}]e^{-i2\pi\alpha t_0} \\
&\quad - \frac{1}{4T_c} Q(f + f_0 + \alpha/2) Q^*(f - f_0 - \alpha/2) \tilde{S}_C^{\alpha+2f_0}(f) \\
&\qquad \times [1 - e^{-i\pi(\alpha+2f_0)T_c}]e^{-i(2\pi[\alpha+2f_0]t_0 + 2\phi_0)} \\
&\quad - \frac{1}{4T_c} Q(f - f_0 + \alpha/2) Q^*(f + f_0 - \alpha/2) \tilde{S}_C^{\alpha-2f_0}(f) \\
&\qquad \times [1 - e^{-i\pi(\alpha-2f_0)T_c}]e^{-i(2\pi[\alpha-2f_0]t_0 - 2\phi_0)}.
\end{aligned}$$

Therefore, for stationary $\{C(nT_s)\}$ and $\{S(nT_s)\}$, the cyclic spectrum for this SQPSK signal is identical to that of BPSK for $\alpha = \pm 2f_0 + k/T_c$, k = odd integers, and for $\alpha = k/T_c$, k = even integers, and is zero for all other values of α.

b) Since the MSK signal has an identical form to that of the SQPSK signal, except for the pulse shape q, then the formula derived in part *a*, with an appropriate replacement for the pulse transform Q, applies to MSK. Consequently, the values of α for which the spectral correlation is non-zero are identical to those for the SQPSK signal.

12.18 Since, from (12.175b), we have the formula

$$A_p(t) = \int_{-\infty}^{\infty} X(t-u)e^{-i2\pi p(t-u)/T} w_T(u)du$$

for the HSR representors, then (with $\alpha = k/T$) the means are given by

$$m_p = E\{A_p(t)\} = \int_{-\infty}^{\infty} m_X(t-u)e^{-i2\pi p(t-u)/T} w_T(u)du$$

$$= \sum_\alpha m_X^\alpha \int_{-\infty}^{\infty} e^{i2\pi(\alpha-p/T)(t-u)} w_T(u)du$$

$$= \sum_\alpha m_X^\alpha W_{1/T}(2\pi[\alpha-p/T])e^{i2\pi(\alpha-p/T)t} = m_X^{p/T},$$

which is independent of t. Also (again using $\alpha = k/T$), the cross-correlations are given by

$$R_{pq}(t+\tau/2, t-\tau/2) = E\{A_p(t+\tau/2)A_q^*(t-\tau/2)\}$$

$$= \iint_{-\infty}^{\infty} R_X(t+\tau/2-u, t-\tau/2-v)e^{-i2\pi(p-q)t/T}e^{-i\pi(p+q)\tau/T}e^{i2\pi(pu-qv)/T}$$

$$\times w_T(u)w_T(v)dudv$$

$$= \sum_\alpha \iint_{-\infty}^{\infty} R_X^\alpha(\tau+v-u)e^{i\pi\alpha(2t-u-v)}e^{-i2\pi(p-q)t/T}e^{-i\pi(p+q)\tau/T}e^{i2\pi(pu-qv)/T}$$

$$\times w_T(u)w_T(v)dudv,$$

$$= \sum_\alpha \int_{-\infty}^{\infty} S_X^\alpha(f)e^{i2\pi f\tau}e^{-i\pi(p+q)\tau/2}\iint_{-\infty}^{\infty} w_T(u)w_T(v)$$

$$\times e^{i2\pi(-f-\alpha/2+p/T)u}e^{-i2\pi(-f+\alpha/2+q/T)v}dudvdfe^{i2\pi(\alpha-\frac{p-q}{T})t}$$

$$= \sum_\alpha \int_{-\infty}^{\infty} S_X^\alpha(f)e^{i2\pi f\tau}W_{1/T}(f+\frac{\alpha}{2}-\frac{p}{T})W_{1/T}(f-\frac{\alpha}{2}-\frac{q}{T})df$$

$$\times e^{i2\pi(\alpha-\frac{p-q}{T})t}e^{-i\pi(p+q)\tau/T}$$

$$= \int_{-\infty}^{\infty} S_X^{(p-q)/T}(f) e^{i2\pi(f - \frac{p+q}{2T})\tau} W_{1/T}(f - \frac{p+q}{2T}) df$$

(since the product of rectangles is non-zero only when $\alpha = (p-q)/T$), which is independent of t. Therefore, the HSR representors $\{A_p(t)\}$ are jointly stationary.

12.19 *a*) Substituting (12.178a) into (12.179b) yields

$$R_X^{n/T}(t-u) = \frac{1}{T} \int_{-T/2}^{T/2} \sum_{p,q=-\infty}^{\infty} R_{pq}(t-u) e^{i2\pi(pt-qu)/T} e^{i2\pi(p-q)v/T}$$

$$\times e^{-i2\pi nv/T} dv e^{-i\pi n(t+u)/T}$$

$$= \sum_{p,q=-\infty}^{\infty} R_{pq}(t-u) e^{i2\pi(pt-qu)/T} \frac{\sin[\pi(p-q-n)]}{\pi(p-q-n)} e^{-i\pi n(t+u)/T}$$

$$= \sum_{p=-\infty}^{\infty} R_{p(p-n)}(t-u) e^{i2\pi p(t-u)/T} e^{-i\pi n(t-u)/T},$$

which is the desired result (12.180) with $\tau = t - u$.

b) Fourier transforming (12.180) immediately yields the result (12.181), which relates the cyclic spectra to the cross-spectra of the HSR representors.

c) Continuing from the solution to exercise 12.18, we have

$$R_{pq}(\tau) = \int_{-\infty}^{\infty} S_X^{(p-q)/T}(f) e^{i2\pi(f - \frac{p+q}{2T})\tau} W_{1/T}(f - \frac{p+q}{2T}) df$$

$$= \int_{-\infty}^{\infty} S_X^{(p-q)/T}(f' + \frac{p+q}{2T}) W_{1/T}(f') e^{i2\pi f'\tau} df' \quad \text{(using } f' = f - (p+q)/2T\text{)}.$$

Fourier transforming both sides yields

$$S_{pq}(f) = S_X^{(p-q)/T}(f + \frac{p+q}{2T}) W_{1/T}(f),$$

which is the desired result (12.184) relating the cross-spectra of the HSR representors to the cyclic spectra.

12.20 Draw a picture showing how (12.187a) interleaves the subsequences from (12.187b).

12.21 For the TSR representors, we have the means

$$E\{Z_p(m)\} = E\{X(mP+p)\} = m_X(mP+p) = m_X(p)$$

for all m since $m_X(\cdot)$ has period P. Thus, the means are constants. The cross-correlations of $Z_p(m)$ and $Z_q(m)$ are given by

$$\begin{aligned}\tilde{R}_{pq}(m+j,\, m) &= E\{Z_p(m+j)Z_q(m)\} = E\{X(mP+jP+p)X(mP+q)\}\\ &= \tilde{R}_X(mP+jP+p,\, mP+q) = \tilde{R}_X(jP+p,\, q)\end{aligned}$$

for all m since $\tilde{R}_X(\cdot,\cdot)$ is jointly periodic with period P. Therefore, the TSR representors $\{Z_p(m)\}$ are jointly stationary (in the wide sense) if $X(t)$ is cyclo-stationary with period P (in the wide sense).

12.22 Substituting (12.196) into (12.197) and the result into (12.198) yields

$$\begin{aligned}\hat{E}\{X(t)\} &= \int_{-\infty}^{\infty} x \frac{d}{dx}\Big(\lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} u[x - x(t-nT)]\Big)dx\\ &= \int_{-\infty}^{\infty} x \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} \delta[x - x(t-nT)]dx\\ &= \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} \int_{-\infty}^{\infty} x\,\delta[x - x(t-nT)]dx\\ &= \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} x(t-nT),\end{aligned}$$

which is the desired result (12.199).

12.23 Substituting (12.199) into (12.200) yields

$$\begin{aligned}\hat{m}_X^\alpha &= \frac{1}{T}\int_{-T/2}^{T/2} \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} x(t+nT)e^{-i2\pi\alpha t}\,dt\\ &= \lim_{N\to\infty} \frac{1}{(2N+1)T} \int_{-(N+1/2)T}^{(N+1/2)T} x(t')e^{-i2\pi\alpha t'}dt', \qquad \alpha = k/T\\ &= \lim_{Z\to\infty} \frac{1}{Z}\int_{-Z/2}^{Z/2} x(t)e^{-i2\pi\alpha t}\,dt, \qquad \alpha = k/T\end{aligned}$$

which agrees with (12.191). Thus, the cyclic mean can be obtained directly from the data as in (12.191) or indirectly from the periodic mean of the data, as in (12.200).

12.24 To relate the correlations of $C(t)$ and $S(t)$ to the cyclic autocorrelations of $X(t)$, we proceed as follows. we have

$$\langle R_C\rangle(\tau) = \langle E\{C(t+\tau/2)C(t-\tau/2)\}\rangle$$

$$= <\frac{1}{2}E\{X(t+\tau/2)X(t-\tau/2)\}[\cos(2\pi\alpha t)+\cos(\pi\alpha\tau)]>$$

$$= \frac{1}{4}<R_X(t+\tau/2,\ t-\tau/2)e^{i2\pi\alpha t}> + \frac{1}{4}<R_X(t+\tau/2,\ t-\tau/2)e^{-i2\pi\alpha t}>$$

$$+ \frac{1}{2}<R_X(t+\tau/2,\ t-\tau/2)>\cos(\pi\alpha\tau)$$

$$= \frac{1}{4}[R_X^{\alpha}(\tau) + R_X^{-\alpha}(\tau)] + \frac{1}{2}<R_X>(\tau)\cos(\pi\alpha\tau)$$

$$= \frac{1}{2}\mathrm{Re}\{R_X^{\alpha}(\tau)\} + \frac{1}{2}<R_X>(\tau)\cos(\pi\alpha\tau)$$

and, similarly,

$$<R_S>(\tau) = -\frac{1}{2}\mathrm{Re}\{R_X^{\alpha}(\tau)\} + \frac{1}{2}<R_X>(\tau)\cos(\pi\alpha\tau).$$

Therefore,

$$\mathrm{Re}\{R_X^{\alpha}(\tau)\} = <R_C>(\tau) - <R_S>(\tau),$$

which is the desired result (12.304a). Likewise, since

$$<R_{CS}>(\tau) = <E\{C(t+\tau/2)S(t-\tau/2)\}>$$

$$= <\frac{1}{2}E\{X(t+\tau/2)X(t-\tau/2)\}[\sin(2\pi\alpha t)-\sin(\pi\alpha\tau)]>$$

$$= \frac{1}{4i}<R_X(t+\tau/2,\ t-\tau/2)e^{i2\pi\alpha t}> - \frac{1}{4i}<R_X(t+\tau/2,\ t-\tau/2)e^{-i2\pi\alpha t}>$$

$$- <R_X(t+\tau/2,\ t-\tau/2)>\sin(\pi\alpha\tau)$$

$$= \frac{1}{4i}[R_X^{-\alpha}(\tau) - R_X^{\alpha}(\tau)] - \frac{1}{2}<R_X>(\tau)\sin(\pi\alpha\tau)$$

$$= -\frac{1}{2}\mathrm{Im}\{R_X^{\alpha}(\tau)\} - \frac{1}{2}<R_X>(\tau)\sin(\pi\alpha\tau)$$

and

$$<R_{SC}>(\tau) = <R_{CS}>(-\tau) = -\frac{1}{2}\mathrm{Im}\{R_X^{\alpha}(\tau)\} + \frac{1}{2}<R_X>(\tau)\sin(\pi\alpha\tau),$$

then

$$\mathrm{Im}\{R_X^{\alpha}(\tau)\} = -[<R_{CS}>(\tau) + <R_{SC}>(\tau)],$$

which is the desired result (12.304b).

12.25 The autocorrelation of the phase-randomized process $Y(t)$ is given by

$$R_Y(t+\tau/2,\ t-\tau/2) = E\{Y(t+\tau/2)Y(t-\tau/2)\} = E\{X(t+\tau/2+\Theta)X(t-\tau/2+\Theta)\}$$

$$= \int_{-\infty}^{\infty} E\{X(t+\tau/2+\theta)X(t-\tau/2+\theta)\} f_\Theta(\theta)d\theta \qquad \text{(using (2.45))}$$

$$= \int_{-\infty}^{\infty} R_X(t+\tau/2+\theta,\, t-\tau/2+\theta) f_\Theta(\theta)d\theta$$

$$= \int_{-\infty}^{\infty} \sum_{\alpha} R_X^\alpha(\tau)e^{i2\pi\alpha(t+\theta)} f_\Theta(\theta)d\theta$$

$$= \sum_{k=-\infty}^{\infty} R_X^{k/T}(\tau)e^{i2\pi kt/T}\Phi_\Theta(2\pi k/T) \qquad (\text{since } \alpha = k/T \text{ only}).$$

Thus, for $Y(t)$ to be WSS, a sufficient condition is that

$$\Phi_\Theta(2\pi k/T) = 0 \qquad \text{for all integers } k \neq 0.$$

This condition is equivalent to

$$\sum_{m=-\infty}^{\infty} \Phi_\Theta(2\pi m/T)\delta(t - m/T) \equiv \delta(t), \qquad -\infty < t < \infty,$$

which has the Fourier transform

$$1 \equiv \sum_{m=-\infty}^{\infty} \Phi_\Theta(2\pi m/T)e^{-i2\pi\nu m/T} = \int_{-\infty}^{\infty} \sum_{m=-\infty}^{\infty} e^{i2\pi\theta m/T}e^{-i2\pi\nu m/T} f_\Theta(\theta)d\theta$$

$$= \int_{-\infty}^{\infty} T\sum_{n=-\infty}^{\infty} \delta(\nu - \theta - nT) f_\Theta(\theta)d\theta = T\sum_{n=-\infty}^{\infty} f_\Theta(\nu - nT), \qquad -\infty < \nu < \infty,$$

which is equivalent to (12.305).

12.26 *a*) Let $U(t)$ and $V(t)$ be defined as in (12.11); for a zero-mean process $X(t)$, we obtain from (2.25) the correlation coefficient (with time-averaging incorporated in the auto-correlations and cross-correlation)

$$\rho = \frac{\langle E\{U(t+\tau/2)V^*(t-\tau/2)\}\rangle}{[\langle E\{U(t+\tau/2)U^*(t-\tau/2)\}\rangle\langle E\{V(t+\tau/2)V^*(t-\tau/2)\}\rangle]^{1/2}}$$

$$= \frac{R_X^\alpha(\tau)}{[\langle R_X\rangle(0)\langle R_X\rangle(0)]^{1/2}} = \frac{R_X^\alpha(\tau)}{\langle R_X\rangle(0)}, \qquad \text{for all } \alpha \text{ and } \tau.$$

b) Integrating (12.34) with respect to f and then applying the Cauchy-Schwarz inequality (11.186) yields

$$\int_{-\infty}^{\infty} |S_X^\alpha(f)|^2 df \le \int_{-\infty}^{\infty} \langle S_X\rangle(f+\alpha/2)\langle S_X\rangle(f-\alpha/2)df \le \int_{-\infty}^{\infty} |\langle S_X\rangle(f)|^2 df.$$

Applying Parseval's relation to this inequality yields the desired inequality (12.307).

12.27 *a*) Since $S(t)$ and $\mathbf{N}(t)$ in the sensor-array model (12.210) are not correlated, then applying the definition of the cyclic correlation to (12.210) yields

$$\begin{aligned}\mathbf{R}_{\mathbf{X}}^{\alpha}(\tau) &= \langle E\{\mathbf{X}(t+\tau/2)\mathbf{X}^{\dagger}(t-\tau/2)\}e^{-i2\pi\alpha t}\rangle \\ &= \langle [E\{S(t+\tau/2)S^{*}(t-\tau/2)\}\mathbf{p}\mathbf{p}^{\dagger} + E\{\mathbf{N}(t+\tau/2)\mathbf{N}^{\dagger}(t-\tau/2)\} \\ &\quad + \mathbf{p}E\{S(t+\tau/2)\mathbf{N}^{\dagger}(t-\tau/2)\} + E\{\mathbf{N}(t+\tau/2)S^{*}(t-\tau/2)\}\mathbf{p}^{\dagger}]e^{-i2\pi\alpha t}\rangle \\ &= R_S^{\alpha}(\tau)\mathbf{p}\mathbf{p}^{\dagger} + \mathbf{R}_{\mathbf{N}}^{\alpha}(\tau),\end{aligned}$$

as desired.

b) Since $\mathbf{N}(t)$ and $\mathbf{M}(t)$ exhibit no cyclostationarity with cycle frequency α and neither is correlated with the other or with $S(t)$, then the cyclic cross-correlation matrix for $X(t)$ and $Y(t)$ in (12.210) ad (12.215) reduces as follows:

$$\begin{aligned}\mathbf{R}_{\mathbf{XY}}^{\alpha}(\tau) &= \langle E\{\mathbf{X}(t+\tau/2)\mathbf{Y}^{\dagger}(t-\tau/2)\}e^{-i2\pi\alpha t}\rangle \\ &= \langle [E\{S(t+\tau/2)S^{*}(t-\tau/2)\}\phi^{*}\mathbf{p}\mathbf{p}^{\dagger} + \mathbf{p}E\{S(t+\tau/2)\mathbf{M}^{\dagger}(t-\tau/2)\} \\ &\quad + E\{\mathbf{N}(t+\tau/2)S^{*}(t-\tau/2)\}\phi^{*}\mathbf{p}^{\dagger} + E\{\mathbf{N}(t+\tau/2)\mathbf{M}^{\dagger}(t-\tau/2)\}]e^{-i2\pi\alpha t}\rangle \\ &= R_S^{\alpha}(\tau)\phi^{*}\mathbf{p}\mathbf{p}^{\dagger}.\end{aligned}$$

c) When (12.212a) is valid, we have

$$\mathbf{R}_X^{\alpha}(\tau) = R_S^{\alpha}(\tau)\mathbf{p}(\theta)\mathbf{p}^{\dagger}(\theta)$$

and

$$\sum_{k=1}^{n} R_{X_k}^{\alpha}(\tau) = \langle E\{\mathbf{X}^{\dagger}(t+\tau/2)\mathbf{X}(t-\tau/2)\}e^{-i2\pi\alpha t}\rangle = R_S^{\alpha}(\tau)\mathbf{p}^{\dagger}(\theta)\mathbf{p}(\theta).$$

Therefore,

$$\int_{-\infty}^{\infty} \mathbf{R}_{\mathbf{X}}^{\alpha}(\tau)\sum_{k=1}^{n} R_{X_k}^{\alpha}(\tau)^{*}\,d\tau = \int_{-\infty}^{\infty} |R_S^{\alpha}(\tau)|^2 d\tau\,[\mathbf{p}^{\dagger}(\theta)\mathbf{p}(\theta)]^{*}\,\mathbf{p}(\theta)\mathbf{p}^{\dagger}(\theta),$$

as desired.

12.28 We can express $X(t)$ as in (12.301a):

$$X(t) = \frac{1}{2}S(t)e^{i2\pi f_0' t} + \frac{1}{2}S^{*}(t)e^{-i2\pi f_0' t}.$$

Then, it follows from (12.302b) and (12.302c) that

$$S_S^{\alpha}(f) = 4S_X^{\alpha}(f+f_0')u(f+f_0'-|\alpha|/2)$$

$$S_{SS^{*}}^{\alpha}(f) = 4S_X^{\alpha+2f_0'}(f)u(\alpha/2+f_0'-|f|).$$

Therefore, for $\alpha = 2f_* = 2(f_0 - f_0')$ we have (with reference to (12.169))

$$S_S^\alpha(f) = 4S_X^{2f_0-2f_0'}(f+f_0')u(f+f_0'-|f_0-f_0'|) \equiv 0 \implies R_S^\alpha \equiv 0$$

(since $\alpha = 2f_0 - 2f_0'$ is not a cycle frequency for the BPSK signal $X(t)$) and

$$S_{SS^*}^\alpha(f) = 4S_X^{2f_0}(f)u(f_0-|f|) \not\equiv 0 \implies R_{SS^*}^\alpha \not\equiv 0$$

(since $\alpha = 2f_0$ is a cycle frequency for the BPSK signal $X(t)$). Also, for $\alpha = 1/T_c$ we have

$$S_S^\alpha(f) = 4S_X^{1/T_c}(f+f_0')u\big(f+f_0'-\frac{1}{2T_c}\big) \not\equiv 0 \implies R_S^\alpha \not\equiv 0$$

(since $\alpha = 1/T_c$ is a cycle frequency of $X(t)$) and

$$S_{SS^*}^\alpha(f) = 4S_X^{2f_0'+1/T_c}(f)u\big(\frac{1}{2T_c}+f_0'-|f|\big) \equiv 0 \implies R_{SS^*}^\alpha \equiv 0$$

(since $\alpha = 2f_0' + 1/T_c$ is not a cycle frequency of $X(t)$).

12.29 The generalized eigenvalues λ for the two matrices $\mathbf{R}$ and $\mathbf{S}$ are the solutions to the equation

$$|\mathbf{R} - \lambda\mathbf{S}| = 0 \quad \text{or} \quad |\mathbf{S}^{-1}\mathbf{R} - \lambda\mathbf{I}| = 0,$$

from which it follows that

$$(a-\lambda)(d-\lambda) - bc = 0$$

or

$$\lambda = \frac{1}{2}(a + d \pm [(a+d)^2 + 4bc - 4ad]^{1/2}) = \frac{a+d}{2} \pm \Big[\frac{a^2+d^2}{4} - \frac{ad}{2} + bc\Big]^{1/2},$$

which is the desired result (12.310). The corresponding generalized eigenvectors $\mathbf{v} = [v_1 \ v_2]^T$ are the solutions to the homogeneous equation (12.308), which is equivalent to

$$(\mathbf{S}^{-1}\mathbf{R} - \lambda\mathbf{I})\mathbf{v} = 0,$$

which reduces to

$$(a-\lambda)v_1 + bv_2 = 0 \quad \text{and} \quad cv_1 + (d-\lambda)v_2 = 0,$$

from which it follows that

$$v_1 = \frac{b}{[b^2 + (\lambda-a)^2]^{1/2}} \quad \text{and} \quad v_2 = \frac{\lambda-a}{b}\,\frac{b}{[b^2+(\lambda-a)^2]^{1/2}}.$$

This is the desired result (12.311).

12.30 The PAM signal is given by (12.134). For a white amplitude sequence $\{A(mT)\}$, we have $\tilde{R}_A(jT) = \tilde{R}_A(0)\delta_j$ and the corresponding cyclic spectrum for the PAM signal is given by (12.143) with $\tilde{S}_A^{\alpha}(f) = 0$ for $\alpha \neq k/T$ and $\tilde{S}_A^{\alpha}(f) = \tilde{R}_A(0)$ for $\alpha = k/T$. Therefore, inverse-Fourier transforming (12.143) yields the cyclic autocorrelation for the PAM signal:

$$\begin{aligned} R_S^{\alpha}(\tau) &= \int_{-\infty}^{\infty} S_S^{\alpha}(f) e^{i2\pi f\tau} df = \frac{1}{T}\tilde{R}_A(0) r_q^{\alpha}(\tau) e^{-i2\pi\alpha t_0} \\ &= \frac{1}{T}\tilde{R}_A(0)\int_{-\infty}^{\infty} q(t+\tau/2)q(t-\tau/2)e^{-i2\pi\alpha t}\, dt e^{-i2\pi\alpha t_0} \\ &= \tilde{R}_A(0)\frac{\sin(\pi\alpha[T-|\tau|])}{\pi\alpha T} e^{-i2\pi\alpha t_0}, \quad \alpha = k/T, \end{aligned}$$

where

$$q(t) \triangleq \begin{cases} 1, & |t| \leq T/2 \\ 0, & |t| > T/2 \end{cases}.$$

Therefore, $|R_S^{\alpha}(\tau)|$ peaks at $\tau = \pm T/2$ when $\alpha = 1/T$. For a QPSK signal, since the sequence $\{C(mT_c)\}$ is white, then $\tilde{R}_C^{\alpha}(jT_c) = \tilde{R}_C^{\alpha}(0)\delta_j$. Therefore, from (12.174) and the preceding result for the PAM signal (with $T = T_c$), we obtain the cyclic autocorrelation for the QPSK signal

$$\begin{aligned} R_S^{\alpha}(\tau) &= \frac{1}{T_c}\tilde{R}_C(0) r_q^{\alpha}(\tau)\cos(2\pi f_0\tau) e^{-i2\pi\alpha t_0} \\ &= \tilde{R}_C(0)\frac{\sin(\pi\alpha[T_c - |\tau|])}{\pi\alpha T_c}\cos(2\pi f_0\tau) e^{-i2\pi\alpha t_0}, \quad \alpha = k/T_c. \end{aligned}$$

When $\alpha = 1/T_c$ and $f_0 T_c = integer$, $|R_S^{\alpha}(\tau)|$ peaks at $\tau = \pm T_c/2$.

12.31 It follows directly from the two-sensor model (12.223) and the assumption that $S(t)$ is not correlated with $N(t)$ and $M(t)$ that the cyclic autocorrelation is given by

$$R_X^{\alpha}(\tau) = \langle E\{X(t+\tau/2)X(t-\tau/2)\}e^{-i2\pi\alpha t}\rangle = R_S^{\alpha}(\tau) + R_N^{\alpha}(\tau)$$

and the cyclic cross-correlation is given by

$$\begin{aligned} R_{YX}^{\alpha}(\tau) &= \langle E\{Y(t+\tau/2)X(t-\tau/2)\}e^{-i2\pi\alpha t}\rangle \\ &= \langle [E\{S(t+\tau/2-\tau_0)S(t-\tau/2)\} + E\{M(t+\tau/2)N(t-\tau/2)\}]e^{-i2\pi\alpha t}\rangle \\ &= \langle R_S(t-\tau_0/2+[\tau-\tau_0]/2,\ t-\tau_0/2-[\tau-\tau_0]/2)e^{-i2\pi\alpha t}\rangle + R_{MN}^{\alpha}(\tau) \\ &= \langle R_S(t'+[\tau-\tau_0]/2,\ t'-[\tau-\tau_0]/2)e^{-i2\pi\alpha t'}\rangle e^{-i\pi\alpha\tau_0} + R_{MN}^{\alpha}(\tau) \\ &= R_S^{\alpha}(\tau-\tau_0)e^{-i\pi\alpha\tau_0} + R_{MN}^{\alpha}(\tau). \end{aligned}$$

Fourier transforming $R_{YX}^{\alpha}(\tau)$ and $R_X^{\alpha}(\tau)$ yields the corresponding auto and cross spectral correlation functions (12.224a) and (12.224b).

12.32 *a)* For the linear combiner, expanding (12.231) with (12.230) substituted in yields

$$\begin{aligned} TAMSE &= \langle E\{|S(t) - \mathbf{g}\mathbf{X}(t)|^2\}\rangle \\ &= \langle E\{|S(t)|^2\}\rangle + \mathbf{g}\langle E\{\mathbf{X}(t)\mathbf{X}^{\dagger}(t)\}\rangle\mathbf{g}^{\dagger} \\ &\quad - \langle E\{S(t)\mathbf{X}^{\dagger}(t)\}\rangle\mathbf{g}^{\dagger} - \mathbf{g}\langle E\{\mathbf{X}(t)S^*(t)\}\rangle \\ &= \langle R_S\rangle + \mathbf{g}\langle\mathbf{R}_{\mathbf{X}}\rangle\mathbf{g}^{\dagger} - \langle\mathbf{R}_{S\mathbf{X}}^T\rangle\mathbf{g}^{\dagger} - \mathbf{g}\langle\mathbf{R}_{S\mathbf{X}}^*\rangle. \end{aligned}$$

Let $\mathbf{g}$ be represented by its real and imaginary parts: $\mathbf{g} = \mathbf{g}_r + i\,\mathbf{g}_i$. Differentiating the TAMSE with respect to the elements of $\mathbf{g}_r$ and $\mathbf{g}_i$, and equating the derivatives to zero yields (in vector form)

$$\frac{\partial}{\partial \mathbf{g}_r} TAMSE = 2\mathrm{Re}\{\mathbf{g}\langle\mathbf{R}_{\mathbf{X}}\rangle\} - 2\mathrm{Re}\{\langle\mathbf{R}_{S\mathbf{X}}^T\rangle\} = \mathbf{0}$$

$$\frac{\partial}{\partial \mathbf{g}_i} TAMSE = -2\mathrm{Im}\{\mathbf{g}\langle\mathbf{R}_{\mathbf{X}}\rangle\} + 2\mathrm{Im}\{\langle\mathbf{R}_{S\mathbf{X}}^T\rangle\} = \mathbf{0},$$

which is equivalent to the orthogonality condition (12.312) and for which $\mathbf{g}_0$ in (12.232) is the solution.

b) It follows from the model (12.210) (and the assumption that $S(t)$ and $\mathbf{N}(t)$ are not correlated) that $\langle\mathbf{R}_{S\mathbf{X}}^T\rangle = \langle R_S\rangle(0)\mathbf{p}^{\dagger}$, which upon substitution into (12.232) yields the simplification (12.235).

c) From the model (12.210) (and the assumption that $S(t)$ and $\mathbf{N(t)}$ are not correlated), we have

$$\begin{aligned} \mathbf{R}_{X_j\mathbf{X}}^{\alpha}(\tau) &= \langle E\{X_j(t+\tau/2)\mathbf{X}^*(t-\tau/2)\}e^{-i2\pi\alpha t}\rangle \\ &= \langle [E\{S(t+\tau/2)S^*(t-\tau/2)\}p_j\mathbf{p}^* + E\{S(t+\tau/2)\mathbf{N}^*(t-\tau/2)\}p_j \\ &\quad + E\{N_j(t+\tau/2)S^*(t-\tau/2)\}\mathbf{p}^* + E\{N_j(t+\tau/2)\mathbf{N}^*(t-\tau/2)\}]e^{-i2\pi\alpha t}\rangle \\ &= R_S^{\alpha}(\tau)p_j\mathbf{p}^* + 0 + 0 + \mathbf{R}_{N_j\mathbf{N}}^{\alpha}(\tau). \end{aligned}$$

Substituting $\mathbf{R}_{X_j\mathbf{X}}^{\alpha}(\tau)$ into (12.237) yields (12.238). Furthermore, for $\mathbf{N}(t)$ exhibiting no cyclostationarity at cycle frequency α, (12.238) reduces to

$$\langle\mathbf{R}_{\tilde{S}\mathbf{X}}\rangle = R_S^{\alpha}(\tau)p_j\mathbf{p}^*\,e^{i\pi\alpha\tau}$$

and, as shown in part *b*,

$$\langle\mathbf{R}_{S\mathbf{X}}\rangle = \langle R_S\rangle(0)\mathbf{p}^{\dagger}.$$

Therefore, the solution (12.232) to the orthogonal condition (12.312), with $S(t)$ replaced by $\tilde{S}(t)$, becomes

$$\mathbf{g}_* = <\mathbf{R}_{\tilde{S}\mathbf{X}}^T><\mathbf{R}_\mathbf{X}>^{-1} = R_S^\alpha(\tau)p_j e^{i\pi\alpha\tau}\mathbf{p}^*<\mathbf{R}_\mathbf{X}>^{-1}$$
$$= \frac{R_S^\alpha(\tau)}{<R_S>(0)} p_j e^{i\pi\alpha\tau}<\mathbf{R}_{\tilde{S}\mathbf{X}}^T><\mathbf{R}_\mathbf{X}>^{-1} = a\,\mathbf{g}_0$$

(where a is defined in (12.241)), as desired.

d) Since, from (12.230), (12.236), and (12.237), we have

$$<R_{\hat{S}\tilde{S}}> = <E\{\hat{S}(t)\tilde{S}^*(t)\}> = <E\{\mathbf{g}\mathbf{X}(t)X_j^*(t+\tau)\}e^{i2\pi\alpha t}>$$
$$= \mathbf{g}\mathbf{R}_{X_j\mathbf{X}}^\alpha(\tau)^* e^{-i\pi\alpha\tau} = \mathbf{g}<\mathbf{R}_{\tilde{S}\mathbf{X}}^*>,$$
$$<R_{\hat{S}}> = <E\{\hat{S}(t)\hat{S}^*(t)\}> = \mathbf{g}<\mathbf{R}_\mathbf{X}>\mathbf{g}^\dagger,$$

and

$$<R_{\tilde{S}}> = <E\{\tilde{S}(t)\tilde{S}^*(t)\}> = <R_{X_j}>,$$

then the squared correlation coefficient (12.243) is given by

$$|\rho|^2 = \frac{|<R_{\hat{S}\tilde{S}}>|^2}{<R_{\hat{S}}><R_{\tilde{S}}>} = \frac{|\mathbf{g}<\mathbf{R}_{\tilde{S}\mathbf{X}}^*>|^2}{<R_{X_j}>\mathbf{g}<\mathbf{R}_\mathbf{X}>\mathbf{g}^\dagger}.$$

It follows from the hint that this quadratic form is maximized by

$$\mathbf{g} = c<\mathbf{R}_{\tilde{S}\mathbf{X}}^T><\mathbf{R}_\mathbf{X}>^{-1} = c\,\mathbf{g}_* = ca\,\mathbf{g}_0,$$

where c is an arbitrary nonzero constant.

12.33 *a)* For the pair of linear combiners, since

$$\hat{S}(t) = \mathbf{g}\mathbf{X}(t) \quad \text{and} \quad \tilde{S}(t) = \mathbf{k}\mathbf{Y}(t)e^{-i2\pi\alpha t},$$

then

$$<R_{\hat{S}\tilde{S}}> = E\{\hat{S}(t)\tilde{S}^*(t)\}> = <\mathbf{g}E\{\mathbf{X}(t)\mathbf{Y}^\dagger(t)\}\mathbf{k}^\dagger e^{i2\pi\alpha t}> = \mathbf{g}\,[\mathbf{R}_{\mathbf{YX}}^\alpha]^\dagger\,\mathbf{k}^\dagger,$$
$$<R_{\hat{S}}> = <E\{\hat{S}(t)\hat{S}^*(t)\}> = \mathbf{g}<\mathbf{R}_\mathbf{X}>\mathbf{g}^\dagger,$$

and

$$<R_{\tilde{S}}> = <E\{\tilde{S}(t)\tilde{S}^*(t)\}> = \mathbf{k}<\mathbf{R}_\mathbf{Y}>\mathbf{k}^\dagger.$$

Substituting these results into the definition of the correlation coefficient magnitude (12.243) yields

$$|\rho|^2 = \frac{|<R_{\hat{S}\tilde{S}}>|^2}{<R_{\hat{S}}><R_{\tilde{S}}>} = \frac{|\mathbf{g}\,[\mathbf{R}_{\mathbf{YX}}^\alpha]^\dagger\mathbf{k}^\dagger|^2}{\mathbf{g}<\mathbf{R}_\mathbf{X}>\mathbf{g}^\dagger\;\mathbf{k}<\mathbf{R}_\mathbf{Y}>\mathbf{k}^\dagger},$$

which reduces to the desired result (12.245) when the optimum steering vector

$$\mathbf{g} = c\,\mathbf{R}_{\tilde{S}\mathbf{X}}^T<\mathbf{R}_\mathbf{X}>^{-1} = c\,\mathbf{k}\mathbf{R}_{\mathbf{YX}}^\alpha<\mathbf{R}_\mathbf{X}>^{-1}$$

is substituted in. (Notice the cancellation of the numerator by one of the factors in the denominator, and compare this with the hint.)

b) Let

$$\mathbf{v}^{\dagger} = \mathbf{k} <\mathbf{R_Y}>^{1/2} \quad \text{and} \quad M = <\mathbf{R_Y}>^{-1/2} \mathbf{R}_{\mathbf{YX}}^{\alpha} <\mathbf{R_X}>^{-1} [\mathbf{R}_{\mathbf{YX}}^{\alpha}]^{\dagger} [<\mathbf{R_Y}>^{-1/2}]^{\dagger}.$$

Then $|\rho|^2$ in (12.245) can be reexpressed in the Rayleigh quotient form (12.313).

c) From (12.314)-(12.315), we obtain (using $\mathbf{v}_i^{\dagger}\mathbf{v}_j = \delta_{ij}$)

$$\mathbf{v}^{\dagger}\mathbf{v} = \sum_{i,j=1}^{n} (a_i \mathbf{v}_i)^{\dagger}(a_j \mathbf{v}_j) = \sum_{i=1}^{n} |a_i|^2,$$

$$\mathbf{v}^{\dagger}\mathbf{M}\mathbf{v} = \sum_{i=1}^{n} (a_i \mathbf{v}_i)^{\dagger} \sum_{l=1}^{n} \lambda_l \mathbf{v}_l \mathbf{v}_l^{\dagger} \sum_{j=1}^{n} (a_j \mathbf{v}_j) = \sum_{i=1}^{n} \lambda_i |a_i|^2,$$

and, therefore,

$$Q = \frac{\mathbf{v}^{\dagger}\mathbf{M}\mathbf{v}}{\mathbf{v}^{\dagger}\mathbf{v}} = \frac{\sum_{i=1}^{n} \lambda_i |a_i|^2}{\sum_{i=1}^{n} |a_i|^2}.$$

Let us use the notation $\sum_{i=1}^{n} |a_i|^2 = A$; then

$$\begin{aligned} Q &= \frac{|a_1|^2}{A}\lambda_1 + \frac{|a_2|^2}{A}\lambda_2 + \cdots + \frac{|a_{n-1}|^2}{A}\lambda_{n-1} + \frac{|a_n|^2}{A}\lambda_n \\ &= \lambda_1 - \frac{|a_2|^2}{A}(\lambda_1 - \lambda_2) - \frac{|a_3|^2}{A}(\lambda_1 - \lambda_3) - \cdots - \frac{|a_n|^2}{A}(\lambda_1 - \lambda_n) \\ &\le \lambda_1 \qquad (\text{since } \lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_n) \end{aligned}$$

with equality if and only if $a_1 \neq 0$ and $a_2 = a_3 = \cdots = a_n = 0$. Thus, Q is maximum when $\{a_i\} = \{a_1, 0, \cdots, 0\}$ for any $a_1 \neq 0$.

12.34 Since, from (12.244) and (12.250), we have

$$\tilde{S}(t) = U(t)e^{-i2\pi\alpha t},$$

then

$$<R_{\hat{S}\tilde{S}}> = <E\{\hat{S}(t)\tilde{S}^*(t)\}> = <E\{U(t)\hat{S}^*(t)\}e^{-i2\pi\alpha t}>^* = <R_{\hat{S}U}^{\alpha}>^*$$

and

$$<R_{\tilde{S}}> = <R_U>.$$

Therefore, from (12.243) we obtain

$$|\rho| = \frac{|<R_{\hat{S}\tilde{S}}>|}{[<R_{\hat{S}}><R_{\tilde{S}}>]^{1/2}} = \frac{|R^{\alpha}_{\hat{S}U}|}{[<R_{\hat{S}}><R_U>]^{1/2}},$$

which is the desired result (12.249).

12.35 The solution is obtained directly by following the hint.

12.36 To optimize the spectral line regenerator, we proceed as follows.

a) Substituting (12.265) into (12.266) and interchanging the order of the double integral and the time-averaging operation yields the desired result (12.318), since

$$<R_X(t-u, t-v)e^{-i2\pi\alpha t}> = R^{\alpha}_X(u-v)e^{-i\pi\alpha(u+v)}.$$

b) When $S(t)$ is absent, then $X(t)$ consists of only stationary WGN $N(t)$, and it follows that

$$R_Y(\tau) = E\{Y(t+\tau)Y(t)\}$$

$$= \iiiint_{-\infty}^{\infty} k(u, v)k(w, z)E\{N(t+\tau-u)N(t+\tau-v)N(t-w)N(t-z)\}dudvdwdz$$

$$= N_0^2\iiiint_{-\infty}^{\infty} k(u, v)k(w, z)[\delta(v-u)\delta(z-w) + \delta(\tau+w-u)\delta(\tau+z-v)$$

$$+ \delta(\tau+z-u)\delta(\tau+w-v)]dudvdwdz$$

$$= N_0^2\Big[\int_{-\infty}^{\infty} k(u, u)du\Big]^2 + 2N_0^2\iint_{-\infty}^{\infty} k(\tau+w, \tau+z)k(w, z)dwdz$$

$$= N_0^2\Big[\int_{-\infty}^{\infty} k(u, u)du\Big]^2 + 2N_0^2\iint_{-\infty}^{\infty} |K(\mu, \nu)|^2 e^{i2\pi\tau(\mu-\nu)}d\mu d\nu,$$

where the convolution theorem for the two-dimensional Fourier transform has been used in the last step. It follows that

$$S_Y(\alpha) = \int_{-\infty}^{\infty} R_Y(\tau)e^{-i2\pi\alpha\tau}d\tau$$

$$= N_0\Big[\int_{-\infty}^{\infty} k(u, u)du\Big]^2\delta(\alpha) + 2N_0^2\iint_{-\infty}^{\infty} |K(\mu, \nu)|^2\delta(\alpha-\mu+\nu)d\mu d\nu$$

$$= 2N_0^2\int_{-\infty}^{\infty} |K(\mu, \mu-\alpha)|^2 d\mu$$

$$= 2N_0^2 \int_{-\infty}^{\infty} |K(f+\alpha/2, f-\alpha/2)|^2 df, \quad \alpha \neq 0,$$

which is the desired result (12.270).

c) By using the Cauchy-Schwarz inequality, we obtain from (12.267)

$$P_Y^\alpha = \left| \int_{-\infty}^{\infty} K(f+\alpha/2, f-\alpha/2) S_X^\alpha(f) df \right|^2$$

$$\leq \int_{-\infty}^{\infty} |K(f+\alpha/2, f-\alpha/2)|^2 df \int_{-\infty}^{\infty} |S_X^\alpha(f)|^2 df$$

with equality if and only if (12.272) holds. Therefore, when $K(f+\alpha/2, f-\alpha/2)$ is given by (12.272), the SNR in (12.271) is maximized.

d) When (12.272) holds, we have

$$P_Y^\alpha = \left| c \int_{-\infty}^{\infty} S_S^\alpha(f)^* S_X^\alpha(f) df \right|^2 = |c|^2 \left[\int_{-\infty}^{\infty} |S_S^\alpha(f)|^2 df \right]$$

(since $S_X^\alpha(f) = S_S^\alpha(f)$ for $\alpha \neq 0$) and

$$S_Y(\alpha) = 2N_0^2 |c|^2 \int_{-\infty}^{\infty} |S_S^\alpha(f)|^2 df.$$

Substituting these results into (12.271) yields the maximum SNR (12.273).

12.37 *a*) For the spectral line regenerator, by using (12.264) for $Y(t)$, we obtain

$$Z(t) = \int_{-\infty}^{\infty} h(w) Y(t-w) dw$$

$$= \iiint_{-\infty}^{\infty} h(w) k(u, v) X(t-w-u) X(t-w-v) du dv dw$$

$$= \iint_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} h(w) k(s-w, r-w) X(t-s) X(t-r) dw \right] ds dr$$

$$= \iint_{-\infty}^{\infty} k'(s, r) X(t-s) X(t-r) ds dr,$$

where $k'(\cdot,\cdot)$ is given by (12.319a). Double Fourier transforming the kernel $k'(\cdot,\cdot)$ yields

$$K'(\mu, \nu) = \iint_{-\infty}^{\infty} k'(u, v) e^{-i2\pi(\mu u - \nu v)} du dv$$

$$= \int_{-\infty}^{\infty} h(s) \iint_{-\infty}^{\infty} k(u-s,\, v-s) e^{-i2\pi(\mu u - \nu v)} du dv\ ds$$

$$= K(\mu,\, \nu) \int_{-\infty}^{\infty} h(s) e^{-i2\pi(\mu-\nu)s} ds \ = K(\mu,\, \nu) H(\mu-\nu),$$

which is the desired result (12.319b). For an ideal bandpass filter with center frequency α and bandwidth $B \to 0$, we have the transfer function

$$H(\mu-\nu) \quad \to \quad \begin{cases} 1, & \mu-\nu=\alpha \\ 0, & \mu-\nu \neq \alpha, \end{cases}$$

from which (12.274a) follows.

b) Since $H(f)$ is centered at $f = \alpha$ and has unity area and a bandwidth of B, then as $B \to 0$, we have

$$H(f) \quad \to \quad \delta(f-\alpha),$$

in which case (12.319b) becomes (12.274b).

c) Substituting (12.272) into (12.274b) yields

$$K'(\mu,\, \nu) = c S_S^{\alpha}(\frac{\mu+\nu}{2})^* \delta(\mu-\nu-\alpha) = c S_S^{\alpha}(\nu+\frac{\alpha}{2})^* \delta(\mu-\nu-\alpha).$$

Inverse Fourier transforming this function yields

$$k'(u,\, v) = \iint_{-\infty}^{\infty} K'(\mu,\, \nu) e^{i2\pi(\mu u - \nu v)} d\mu d\nu = c \int_{-\infty}^{\infty} S_S^{\alpha}(\nu+\alpha/2)^* e^{i2\pi(u-v)\nu} d\nu e^{i2\pi\alpha u}$$

$$= c R_S^{\alpha}(u-v)^* e^{i\pi\alpha(u+v)},$$

which is the desired result (12.320). Substituting (12.320) (with $c = 1/N_0^2 W$) into (12.264) (with limits of integration reduced to $\pm W/2$) yields

$$Y(t) = \frac{1}{N_0^2 W} \iint_{-W/2}^{W/2} R_S^{\alpha}(u-v)^* e^{i\pi\alpha(u+v)} X(t-u) X(t-v) du dv$$

$$= \frac{1}{N_0^2 W} \int_{-W}^{W} R_S^{\alpha}(\tau)^* \int_{-(W-|\tau|)/2}^{(W-|\tau|)/2} X(t+\sigma+\tau/2) X(t+\sigma-\tau/2) e^{-i2\pi\alpha\sigma} d\sigma d\tau$$

$$= \frac{1}{N_0^2} \int_{-\infty}^{\infty} R_S^{\alpha}(\tau)^* R_X^{\alpha}(t,\, \tau)_W d\tau e^{i2\pi\alpha t}$$

$$= \frac{1}{N_0^2} \int_{-\infty}^{\infty} S_S^{\alpha}(f)^* P_X^{\alpha}(t,\, f)_W df e^{i2\pi\alpha t} \qquad \text{(by Parseval's relation)},$$

which is the desired result (12.275) for the single-cycle detector.

12.38 *a*) From (12.278), we obtain

$$E\{\hat{S}(t)X(v)\} = \int_{-\infty}^{\infty} h_0(t, u)E\{X(u)X(v)\}du = \int_{-\infty}^{\infty} h_0(t, u)R_X(u, v)du$$

and

$$E\{S(t)X(v)\} = R_{SX}(t, v).$$

Substituting these equations into (12.280) yields the desired result (12.281) for the optimum time-variant filter.

b) From (12.279), we obtain

$$TAMSE_{\min} = <E\{[S(t)-\hat{S}(t)]S(t)\}> - <E\{[S(t)-\hat{S}(t)]\hat{S}(t)\}>$$

$$= <E\{[S(t)-\hat{S}(t)]S(t)\} - \int_{-\infty}^{\infty} h_0(t, u)E\{[S(t)-\hat{S}(t)]X(u)\}du>$$

$$= <E\{[S(t)-\hat{S}(t)]S(t)\}> \quad \text{(using (12.280))}$$

$$= <E\{S^2(t)\} - E\{\hat{S}(t)S(t)\}>,$$

which is (12.322). Substituting (12.278) now yields

$$TAMSE_{\min} = <R_S(t, t) - \int_{-\infty}^{\infty} h_0(t, u)E\{S(t)X(u)\}du>$$

$$= <R_S(t, t) - \int_{-\infty}^{\infty} h_0(t, u)R_{SX}(t, u)du>,$$

which is the desired optimum-performance formula (12.282).

12.39 *a*) Substituting (12.283a) and (12.4) into the right-hand side of (12.281) yields

$$R_{SX}(t, v) = \sum_{\beta,\alpha} \int_{-\infty}^{\infty} g_\beta(t-u)R_X^\alpha(v-u)e^{i2\pi\beta u}e^{i\pi\alpha(v+u)}du$$

$$= \sum_{\beta,\alpha} \int_{-\infty}^{\infty} g_\beta(w)R_X^\alpha(v-t+w)e^{i2\pi\beta(t-w)}e^{i\pi\alpha(v+t-w)}dw \quad \text{(using } t-u=w\text{)}.$$

The cyclic cross-correlation of $S(t)$ and $X(t)$ is given by

$$R_{SX}^\gamma(\tau) = <R_{SX}(z+\tau/2, z-\tau/2)e^{-i2\pi\gamma z}>$$

$$= \sum_{\beta,\alpha} \int_{-\infty}^{\infty} g_\beta(w)R_X^\alpha(\tau-w) <e^{i2\pi(\alpha+\beta-\gamma)z}> e^{-i\pi(2\beta+\alpha)w}dwe^{i\pi\beta\tau}$$

$$= \sum_{\beta} \int_{-\infty}^{\infty} g_{\beta}(w) R_X^{\gamma-\beta}(\tau - w) e^{i\pi(\beta+\gamma)(\tau-w)} dw e^{-i\pi\gamma\tau},$$

which can be reexpressed as

$$R_{SX}^{\gamma}(\tau) e^{i\pi\gamma\tau} = \sum_{\beta} g_{\beta}(\tau) \otimes [R_X^{\gamma-\beta}(\tau) e^{i\pi(\gamma+\beta)\tau}],$$

which is the desired design equation (12.284).

Substituting (12.283a) and the Fourier series expansion of $R_{SX}(t, u)$ into (12.282) yields

$$TAMSE_{\min} = \langle R_S(t, t) \rangle - \langle \int_{-\infty}^{\infty} \sum_{\alpha,\beta} g_{\beta}(t-u) e^{i2\pi\beta u} R_{SX}^{\alpha}(t-u) e^{i\pi\alpha(t+u)} du \rangle$$

$$= R_S^0(0) - \langle \int_{-\infty}^{\infty} \sum_{\alpha,\beta} g_{\beta}(w) R_{SX}^{\alpha}(w) e^{-i\pi(2\beta+\alpha)w} dw e^{i2\pi(\beta+\alpha)t} \rangle$$

$$= R_S^0(0) - \int_{-\infty}^{\infty} \sum_{\alpha,\beta} g_{\beta}(w) R_{SX}^{\alpha}(w) e^{-i\pi(2\beta+\alpha)w} dw \langle e^{i2\pi(\beta+\alpha)t} \rangle$$

$$= R_S^0(0) - \sum_{\beta} \int_{-\infty}^{\infty} g_{\beta}(w) R_{SX}^{-\beta}(w) e^{-i\pi\beta w} dw \qquad (\text{using } \langle e^{i2\pi(\beta+\alpha)t} \rangle = \delta_{\alpha+\beta})$$

$$= R_S^0(0) - \sum_{\beta} \int_{-\infty}^{\infty} g_{\beta}(\tau) R_{SX}^{\beta}(\tau)^* e^{-i\pi\beta\tau} d\tau,$$

which is the desired optimum-performance formula (12.285).

b) Fourier transforming both sides of (12.284) and using the convolution theorem yields the frequency-domain design equation (12.286) immediately. Applying Parseval's relation to (12.285) yields immediately the frequency-domain optimum-performance formula (12.288):

$$TAMSE_{\min} = \int_{-\infty}^{\infty} S_S^0(f) df - \sum_{\beta} \int_{-\infty}^{\infty} G_{\beta}(f) S_{SX}^{\beta}(f - \beta/2)^* df$$

$$= \int_{-\infty}^{\infty} [S_S^0(f) - \sum_{\beta} G_{\beta}(f) S_{SX}^{\beta}(f - \beta/2)^*] df.$$

12.40 For the system-identification problem, since $N(t)$ and $M(t)$ are statistically independent of $X(t)$ and $Y(t)$ and can be assumed to have zero mean values, then

$$S_{X'Y'}^{\alpha}(f) = S_{XY}^{\alpha}(f) + S_{NM}^{\alpha}(f)$$

$$S_{X'}^{\alpha}(f) = S_X^{\alpha}(f) + S_N^{\alpha}(f).$$

Hence, it follows that

$$\hat{H}(f) = \frac{S_{XY}^{\alpha}(f+\alpha/2) + S_{NM}^{\alpha}(f+\alpha/2)}{S_X^{\alpha}(f+\alpha/2) + S_N^{\alpha}(f+\alpha/2)} = \frac{H(f) + S_{NM}^{\alpha}(f+\alpha/2)/S_X^{\alpha}(f+\alpha/2)}{1 + S_N^{\alpha}(f+\alpha/2)/S_X^{\alpha}(f+\alpha/2)}.$$

If $N(t)$ and $M(t)$ exhibit no cyclostationarity at $\alpha = \alpha_0$, we then obtain the perfect transfer function approximation:

$$\hat{H}(f) = \frac{H(f) + 0/S_X^{\alpha}(f+\alpha/2)}{1 + 0/S_X^{\alpha}(f+\alpha/2)} = H(f);$$

whereas for $\alpha = 0$, we have the imperfect approximation

$$\hat{H}(f) = \frac{H(f) + S_{NM}^{0}(f)/S_X^{0}(f)}{1 + S_N^{0}(f)/S_X^{0}(f)} \neq H(f).$$

12.41 We obtain the cyclic autocorrelation for the product $Z(t)$ of time-series $X(t)$ and $Y(t)$ by definition

$$\begin{aligned} R_Z^{\alpha}(\tau) &= \langle E\{Z(t+\tau/2)Z(t-\tau/2)\}e^{-i2\pi\alpha t}\rangle \\ &= \langle E\{X(t+\tau/2)X(t-\tau/2)\}E\{Y(t+\tau/2)Y(t-\tau/2)\}e^{-i2\pi\alpha t}\rangle \\ &= \langle R_X(t+\tau/2,\, t-\tau/2)R_Y(t+\tau/2,\, t-\tau/2)e^{-i2\pi\alpha t}\rangle. \end{aligned}$$

Substituting (12.4) now yields

$$\begin{aligned} R_Z^{\alpha}(\tau) &= \langle \sum_{\beta} R_X^{\beta}(\tau)e^{i2\pi\beta t}\sum_{\gamma} R_Y^{\gamma}(\tau)e^{i2\pi\gamma t}\, e^{-i2\pi\alpha t}\rangle \\ &= \sum_{\beta,\gamma} R_X^{\beta}(\tau)R_Y^{\gamma}(\tau)\langle e^{i2\pi(\beta+\gamma-\alpha)t}\rangle = \sum_{\beta} R_X^{\beta}(\tau)R_Y^{\alpha-\beta}(\tau). \end{aligned}$$

Fourier transforming $R_Z^{\alpha}(\tau)$ and applying the time-frequency dual of the convolution theorem yields the desired result (12.323):

$$\begin{aligned} S_Z^{\alpha}(f) &= \sum_{\beta} S_X^{\beta}(f)\otimes S_Y^{\alpha-\beta}(f) = \sum_{\beta}\int_{-\infty}^{\infty} S_X^{\beta}(\nu)S_Y^{\alpha-\beta}(f-\nu)\,d\nu \\ &= \sum_{\beta}\int_{-\infty}^{\infty} S_X^{\alpha-\beta}(f-\nu)S_Y^{\beta}(\nu)\,d\nu \end{aligned}$$

(using the fact that $A \otimes B = B \otimes A$ for both discrete and continuous convolution).

Chapter 13

Minimum-Mean-Squared-Error Estimation

13.1 All vectors $X = \{\alpha_1, \alpha_2, \alpha_3\}$ that lie on a line through the origin of 3-dimensional Euclidean space can be expressed as $X = \beta Y$, where Y is any vector on this line, and β is an appropriate scalar. It follows that the set of such vectors on a line is a linear space since every linear combination of every pair of vectors X_1 and X_2 on this line is another vector on this line:

$$\sigma X_1 + \gamma X_2 = \sigma\beta_1 Y + \gamma\beta_2 Y = (\sigma\beta_1 + \gamma\beta_2)Y \triangleq \beta_3 Y = X_3.$$

However, if the line does not pass through the origin, then all vectors on the line are of the form

$$X = \beta Y + Z$$

for fixed vectors Y and Z, where $Z \neq 0$ is orthogonal to Y (Z can be thought of as giving the magnitude and direction of the line segment that is perpendicular to the line of interest and connects the origin to this line. It follows that the set of vectors on this line is not closed under linear combination

$$\sigma X_1 + \gamma X_2 = (\sigma\beta_1 + \gamma\beta_2)Y + (\sigma+\gamma)Z \neq \beta_3 Y + Z$$

(the inequality holds for all σ and γ except the special case where $\sigma + \gamma = 1$). Thus, lines that do not pass through the origin are not linear spaces.

Similar arguments can be construed to verify that all planes through the origin are linear spaces, and no plane that does not pass through the origin is a linear space.

To prove that no set of vectors other than those that form lines and planes can be a linear subspace of 3 dimensional Euclidean space requires more work. For example, this can be done by contradiction; that is, by proving that if a set of vectors is closed under linear combination, then it must be either a line or plane through the origin (or all of 3 dimensional Euclidean space).

13.2 Let the basis vectors in the given subspace be

$$Y_1 = \{1, -1, 0\} \quad \text{and} \quad Y_2 = \{0, 1, 1\}.$$

Then the estimate can be expressed as

$$\hat{X} = h_1 Y_1 + h_2 Y_2 = \{h_1, h_2 - h_1, h_2\}.$$

The solution to (13.19a) for the optimum 2-tuple of scalars $H_0 = [h_1^0 \quad h_2^0]^T$ is given by (using $X = \{\alpha_1, \alpha_2, \alpha_3\}$)

$$\begin{bmatrix} h_1^0 \\ h_2^0 \end{bmatrix} = \begin{bmatrix} (Y_1, Y_1) & (Y_2, Y_1) \\ (Y_1, Y_2) & (Y_2, Y_2) \end{bmatrix}^{-1} \begin{bmatrix} (X, Y_1) \\ (X, Y_2) \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}^{-1} \begin{bmatrix} \alpha_1 - \alpha_2 \\ \alpha_2 + \alpha_3 \end{bmatrix}$$

$$= \begin{bmatrix} 2/3 & 1/3 \\ 1/3 & 2/3 \end{bmatrix} \begin{bmatrix} \alpha_1 - \alpha_2 \\ \alpha_2 + \alpha_3 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 2\alpha_1 - \alpha_2 + \alpha_3 \\ \alpha_1 + \alpha_2 + 2\alpha_3 \end{bmatrix}$$

Therefore,

$$\hat{X}_0 = h_1^0 Y_1 + h_2^0 Y_2 = \frac{1}{3} \begin{bmatrix} 2\alpha_1 - \alpha_2 + \alpha_3 \\ -\alpha_1 + 2\alpha_2 + \alpha_3 \\ \alpha_1 + \alpha_2 + 2\alpha_3 \end{bmatrix}$$

13.3 Using the inner product definition (13.8c) with $T = [0, \ 1]$ yields

$$(X, Y) = \int_T X(t)Y(t)dt = \int_0^1 \cos(2\pi t)\sin(2\pi t)dt = \frac{1}{4\pi}\sin^2(2\pi t)\Big|_0^1 = 0.$$

Therefore, X and Y are orthogonal for $T = [0, \ 1]$. However, if $T = [a, \ b]$, then

$$(X, Y) = \int_T X(t)Y(t)dt = \frac{1}{4\pi}\sin^2(2\pi t)\Big|_a^b$$

$$= \frac{\sin^2(2\pi b)}{4\pi} - \frac{\sin^2(2\pi a)}{4\pi} = \frac{1}{8\pi}[\cos(4\pi a) - \cos(4\pi b)] = 0$$

if and only if $b - a = k/2$ for some Integer k.

13.4 Let the basis functions $Y_1(t)$ and $Y_2(t)$ be

$$Y_1(t) = \cos(t) \qquad \text{and} \qquad Y_2(t) = \sin(t).$$

Then, the estimate (orthogonal projection) can be expressed as

$$\hat{X}(t) = \sigma Y_1(t) + \gamma Y_2(t) \quad \text{for} \quad t \in [0, \ \pi].$$

Since the function to be estimated is given by

$$X(t) = \begin{cases} 1, & 0 \le t \le \pi \\ 0, & \textit{otherwise}, \end{cases}$$

then the inner products of interest in the equation (13.19a) are given by

$$(X, Y_1) = \int_0^\pi \cos(t)dt = 0, \qquad (X, Y_2) = \int_0^\pi \sin(t)dt = 2$$

$$(Y_1, Y_1) = \int_0^{\pi} \cos^2(t)dt = \pi/2, \qquad (Y_2, Y_2) = \int_0^{\pi} \sin^2(t)dt = \pi/2$$

$$(Y_1, Y_2) = (Y_2, Y_1) = \int_0^{\pi} \cos(t)\sin(t)dt = 0.$$

Substituting these results into (13.19a) yields

$$\begin{bmatrix} \pi/2 & 0 \\ 0 & \pi/2 \end{bmatrix} \begin{bmatrix} h_1^0 \\ h_2^0 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

which has the solution

$$h_1^0 = \sigma^0 = 0 \quad \text{and} \quad h_2^0 = \gamma^0 = \frac{4}{\pi}.$$

Therefore, the orthogonal projection of $X(t)$ is

$$\hat{X}_0(t) = \sigma^0\cos(t) + \gamma^0\sin(t) = \frac{4}{\pi}\sin(t).$$

13.5 To verify Woodbury's identity, we multiply both sides of (13.266) by $[\mathbf{A} + \mathbf{bc}]$ to obtain

$$\mathbf{I} = [\mathbf{A} + \mathbf{bc}][\mathbf{A}^{-1} - \frac{\mathbf{A}^{-1}\mathbf{bcA}^{-1}}{1 + \mathbf{cA}^{-1}\mathbf{b}}] = \mathbf{I} - \frac{\mathbf{bcA}^{-1}}{1 + \mathbf{cA}^{-1}\mathbf{b}} + \mathbf{bcA}^{-1} - \frac{\mathbf{bcA}^{-1}\mathbf{bcA}^{-1}}{1 + \mathbf{cA}^{-1}\mathbf{b}}$$

$$= \mathbf{I} + \frac{\mathbf{bcA}^{-1}}{1 + \mathbf{cA}^{-1}\mathbf{b}}\mathbf{cA}^{-1}\mathbf{b} - \frac{\mathbf{b}(\mathbf{cA}^{-1}\mathbf{b})\mathbf{cA}^{-1}}{1 + \mathbf{cA}^{-1}\mathbf{b}} = \mathbf{I} \qquad (\text{since } \mathbf{cA}^{-1}\mathbf{b} \text{ is a scalar}),$$

which is indeed an identity.

13.6 To verify (13.45b), we multiply both sides of (13.39) by $\frac{1}{l}Y(l)$ to obtain

$$\frac{1}{l}\bar{R}_Y^{-1}(l)Y(l) = \frac{1}{l-1}[R_Y^{-1}(l-1)Y(l) - \frac{\bar{R}_Y^{-1}(l-1)Y(l)Y^T(l)\bar{R}_Y^{-1}(l-1)Y(l)}{(l-1) + Y^T(l)\bar{R}_Y^{-1}(l-1)Y(l)}]$$

$$= \frac{1}{l-1}\frac{R_Y^{-1}(l-1)Y(l)(l-1) + 0}{(l-1) + Y^T(l)\bar{R}_Y^{-1}(l-1)Y(l)} = \frac{R^{-1}(l-1)Y(l)}{(l-1) + Y^T(l)\bar{R}_Y^{-1}(l-1)Y(l)},$$

which is the desired result (13.45b).

13.7 Since, from (13.33) and (13.34), we have (using definitions (13.24) and (13.36))

$$\bar{R}_{xY}(l) = \frac{1}{l}\sum_{n=1}^{l} x_n Y(n) = \frac{1}{l}[Y(1) \;\; Y(2) \cdots Y(l)]X = \frac{1}{l}M^T X$$

$$\bar{R}_Y(l) = \frac{1}{l}\sum_{n=1}^{l} Y(n)Y^T(n) = \frac{1}{l}[Y(1)\ Y(2)\cdots Y(l)]\begin{bmatrix} Y^T(1) \\ Y^T(2) \\ \cdot \\ \vdots \\ \cdot \\ Y^T(l) \end{bmatrix} = \frac{1}{l}M^T M,$$

then substituting the above results into (13.32) yields (using the definition (13.268))

$$H_0(l) = \bar{R}_Y^{-1}(l)\bar{R}_{xY}(l) = [\frac{1}{l}M^T M]^{-1}\frac{1}{l}M^T X = M^{(-1)}X,$$

which is (13.267), as desired.

13.8 Since

$$E\{[X \pm Y]^2\} = E\{X^2\} \pm 2E\{XY\} + E\{Y^2\},$$

then we have

$$E\{XY\} = 0 \Longleftrightarrow E\{[X \pm Y]^2\} = E\{X^2\} + E\{Y^2\},$$

which (using definitions (13.8d) and (13.9d)) is (13.12), as desired. We also have

$$E\{(X+Y)^2 + (X-Y)^2\} = E\{X^2 + 2XY + Y^2 + X^2 - 2XY + Y^2\} = 2E\{X^2\} + 2E\{Y^2\},$$

which (using definitions (13.8d) and (13.9d)) is (13.13), as desired. Furthermore, we have

$$E\{(X-Y)^2\} = E\{X^2 - 2XY + Y^2\} = E\{X^2\} + E\{Y^2\} - 2E\{XY\},$$

which (using definitions (13.8d), (13.9d) and (13.10)) is (13.14), as desired. Finally, we have proved in exercise 2.14 that

$$E\{XY\} \le \sqrt{E\{X^2\}E\{Y^2\}},$$

which (using definition (13.8d) and (13.9d)) is (13.15), as desired.

13.9 *a*) The *MSE* can be expressed as

$$\begin{aligned} MSE &= E\{(X-\hat{X})^2\} = E\{[(X-\hat{X}_0) + (\hat{X}_0-\hat{X})]^2\} \\ &= E\{(X-\hat{X}_0)^2\} + 2E\{(X-\hat{X}_0)(\hat{X}_0-\hat{X})\} + E\{(\hat{X}_0-\hat{X})^2\}. \end{aligned}$$

b) Since $\hat{X} = g(Y)$ and $\hat{X}_0 = g_0(Y) = E\{X|Y\}$, then

$$E\{(X-\hat{X}_0)g(Y)\} = E\{(X-E\{X|Y\})g(Y)\} = 0 \qquad \text{for all} \qquad g(\cdot)$$

and, therefore,

$$\begin{aligned} E\{(X-\hat{X}_0)(\hat{X}_0-\hat{X})\} &= E\{(X-E\{X|Y\})g_0(Y)\} - E\{(X-E\{X|Y\})g(Y)\} \\ &= 0 - 0 = 0. \end{aligned}$$

Thus,

$$MSE = E\{(X-\hat{X}_0)^2\} + E\{(\hat{X}_0-\hat{X})^2\},$$

which is minimized with respect to $\hat{X}$ if and only if $\hat{X} = \hat{X}_0 = E\{X|Y\}$.

13.10 (i) From the orthogonality condition, we have

$$E\{(X-\hat{X}_0)\hat{X}_0\} = 0 \quad \text{or} \quad E\{X\hat{X}_0\} = E\{\hat{X}_0^2\}.$$

Therefore,

$$\begin{aligned} MSE_0 &= E\{(X-\hat{X}_0)^2\} = E\{(X-\hat{X}_0)X\} - E\{(X-\hat{X}_0)\hat{X}_0\} \\ &= E\{(X-\hat{X}_0)X\} = E\{X^2\} - E\{X\hat{X}_0\} \\ &= E\{X^2\} - E\{(X-\hat{X}_0)\hat{X}_0 + \hat{X}_0\hat{X}_0\} \\ &= E\{X^2\} - E\{(X-\hat{X}_0)\hat{X}_0\} - E\{\hat{X}_0^2\} = E\{X^2\} - E\{\hat{X}_0^2\}, \end{aligned}$$

which verifies (13.66).

(ii) It follows from the orthogonality condition (13.72) that

$$MSE_0 = E\{X^2\} - E\{X\hat{X}_0\}$$

(as in (13.66)). Substituting (13.80) into (13.70) and the result into this expression for MSE_0 yields

$$MSE_0 = R_X - E\{X\mathbf{Y}^T\mathbf{h}_0\} = R_X - \mathbf{R}_{X\mathbf{Y}}^T\mathbf{R}_{\mathbf{Y}}^{-1}\mathbf{R}_{X\mathbf{Y}} = R_X(1-\rho^2),$$

where

$$\rho = \frac{\mathbf{R}_{X\mathbf{Y}}^T\mathbf{R}_{\mathbf{Y}}^{-1}\mathbf{R}_{X\mathbf{Y}}}{R_X}.$$

This is the desired result (13.83)-(13.84).

(iii) It follows from the orthogonality condition (13.88) that

$$MSE_0 = E\{X^2\} - E\{X\hat{X}_0\}$$

(as in (13.66)). Substituting (13.86) into this expressing yields

$$MSE_0 = R_X - \int_V E\{XY(t)\}h_0(t)dt = R_X - \int_V R_{XY}(t)h_0(t)dt = R_X(1-\rho^2),$$

where

$$\rho^2 = \frac{1}{R_X}\int_V R_{XY}(t)h_0(t)dt,$$

as desired.

13.11 From the result of exercise 1.12*d*, we have

$$E\{X|y\} = \beta'' = \gamma\frac{\alpha}{\alpha'}y + \beta - \gamma\frac{\alpha}{\alpha'}\beta' = h_0 y + f_0,$$

where

$$h_0 = \gamma\frac{\alpha}{\alpha'}, \quad f_0 = \beta - \gamma\frac{\alpha}{\alpha'}\beta',$$

$$\beta = E\{X\}, \quad \beta' = E\{Y\}, \quad \alpha = \sqrt{Var\{X\}}, \quad \alpha' = \sqrt{Var\{Y\}},$$

and

$$\gamma = \frac{Cov\{X, Y\}}{\alpha\alpha'}.$$

If $\beta = E\{X\} = 0$ and $\beta' = E\{Y\} = 0$, then $f_0 = 0$.

13.12 (i) Substituting $\hat{Z} = h_1 Y + f_1$ into the given orthogonality condition yields

$$\begin{aligned} E\{(\hat{Z}-Z)(hY+f)\} &= E\{(\hat{Z}-Z)Y\}h - E\{\hat{Z}-Z\}f \\ &= (h_1[\sigma_Y^2+m_Y^2] + m_Y f_1 - E\{ZY\})h - (h_1 m_Y + f_1 - E\{Z\})f \\ &= 0 \qquad \text{for all } h, f. \end{aligned}$$

Thus, the coefficients of h and f must be zero:

$$0 = h_1[\sigma_Y^2+m_Y^2] + m_Y f_1 - \int_{-\infty}^{\infty} y f_{X|Y}(x|y) f_Y(y)\,dy \quad (\text{using } Z = f_{X|Y}(x|Y))$$

$$= h_1[\sigma_Y^2+m_Y^2] + m_Y f_1 - \int_{-\infty}^{\infty} y f_{Y|X}(y|x) f_X(x)\,dy$$

$$= h_1[\sigma_Y^2+m_Y^2] + m_Y f_1 - f_X(x) E\{Y|x\}$$

and

$$0 = h_1 m_Y + f_1 - E\{Z\} = h_1 m_Y + f_1 - \int_{-\infty}^{\infty} f_{X|Y}(x|y) f_Y(y)\,dy$$

$$= h_1 m_Y + f_1 - f_X(x).$$

Solving these two equations for h_1 and f_1 yields

$$h_1 = \frac{1}{\sigma_Y^2}(E\{Y|x\} - m_Y) f_X(x)$$

$$f_1 = \frac{1}{\sigma_Y^2}(\sigma_Y^2 + m_Y^2 - E\{Y|x\} m_Y) f_X(x).$$

Therefore, the optimum linear estimate is given by

$$\hat{Z} = h_1 Y + f_1 = \frac{f_X(x)}{\sigma_Y^2}[(E\{Y|x\} - m_Y)Y + \sigma_Y^2 + m_Y^2 - E\{Y|x\}m_Y]$$

$$= f_X(x)[1 + \frac{E\{Y|x\} - m_Y}{\sigma_Y^2}(Y - m_Y)],$$

which is the desired result (13.270).

(ii) Substituting (13.270) into (13.271) yields

$$\hat{E}\{X|Y\} = \int_{-\infty}^{\infty} x\hat{f}_{X|Y}(x|Y)dx$$

$$= \int_{-\infty}^{\infty} xf_X(x)dx + \frac{1}{\sigma_Y^2}\int_{-\infty}^{\infty} xf_X(x)[E\{Y|x\} - m_Y]dx\,(Y - m_Y)$$

$$= m_X + \frac{1}{\sigma_Y^2}[\iint_{-\infty}^{\infty} xyf_X(x)f_{Y|X}(y|x)dxdy - m_X m_Y](Y - m_Y)$$

$$= m_X + \frac{K_{XY}}{\sigma_Y^2}(Y - m_Y),$$

which is the desired result (13.272).

(iii) The generalized linear MMSE estimate of X based on the observation Y is given by

$$\hat{X} = E\{X|Y\} = h_0 Y + f_0,$$

where h_0 and f_0 satisfy the orthogonality condition

$$E\{(\hat{X} - X)(hY + f)\} = 0 \quad \text{for all } h, f.$$

Substituting $\hat{X} = h_0 Y + f_0$ into this equation leads to the pair of equations

$$h_0(\sigma_Y^2 + m_Y^2) + f_0 m_Y - E\{XY\} = 0$$

$$h_0 m_Y + f_0 - m_X = 0,$$

which have the solution

$$h_0 = \frac{K_{XY}}{\sigma_Y^2} \quad \text{and} \quad f_0 = m_X - \frac{K_{XY}}{\sigma_Y^2}m_Y.$$

It can be seen that this generalized MMSE estimate $\hat{X}$ is equivalent to the wide-sense conditional mean $\hat{E}\{X|Y\}$.

(iv) If X and Y are jointly Gaussian, then we obtain from exercise 13.11 the conditional mean

$$E\{X|Y\} = \gamma\frac{\alpha}{\alpha'}Y + \beta - \gamma\frac{\alpha}{\alpha'}\beta' = \frac{K_{XY}}{\sigma_Y^2}Y + m_X - \frac{K_{XY}}{\sigma_Y^2}m_Y$$

$$= m_X + \frac{K_{XY}}{\sigma_Y^2}(Y - m_Y),$$

which is identical to the wide-sense conditional mean $\hat{E}\{X|Y\}$.

13.13 For zero-mean jointly Gaussian variables X and Y, we have $E\{X|Y\} = h_0 Y$ and $h_0 = K_{XY}/\sigma_Y^2$. Therefore, the left-hand side of (13.275) is equal to

$$L \triangleq E\{(X - h_0 Y)^2|y\} = E\{X^2|y\} - 2h_0 E\{X|y\}y + h_0^2 y^2 = E\{X^2|y\} - h_0^2 y^2,$$

whereas the right-hand side is equal to

$$\begin{aligned} R &= E\{(X - h_0 Y)^2\} = E\{X^2\} - 2E\{XE\{X|Y\}\} + E\{(E\{X|Y\})^2\} \\ &= E\{X^2\} - 2h_0 E\{XY\} + h_0^2 E\{Y^2\} \\ &= E\{X^2\} - 2K_{XY}^2/\sigma_Y^2 + K_{XY}^2/\sigma_Y^2 \\ &= E\{X^2\} - K_{XY}^2/\sigma_Y^2 = \sigma_X^2 - K_{XY}^2/\sigma_Y^2. \end{aligned}$$

However, since

$$E\{X^2|y\} = \int_{-\infty}^{\infty} x^2 f_{X|Y}(x|y)dx = \sigma_{X|y}^2 + (E\{X|y\})^2 = \sigma_{X|y}^2 + (h_0 y)^2,$$

then $L = \sigma_{X|y}^2$ and, from exercise 1.12*d* (with X and Y interchanged), we have

$$L = \sigma_{X|y}^2 = \sigma_X^2(1-\rho^2) = \sigma_X^2 - K_{XY}^2/\sigma_Y^2.$$

Therefore, $L = R$, as desired. (Note: this certainly is not valid, in general, for non-Gaussian variables.)

13.14 *a*) Substituting the estimate (13.95) into the orthogonality condition (13.94) yields

$$E\{(X - \hat{X}_0)Y(v)\} = E\{XY(v)\} - \int_V h_0(u)E\{Y(u)Y(v)\}du = 0 \quad \text{for all } v \in V$$

or, equivalently,

$$E\{XY(v)\} = \int_V h_0(u)E\{Y(u)Y(v)\}du \qquad \text{for all } v \in V,$$

which is the design equation (13.96), as desired.

b) Adding and subtracting $\hat{X}_0$ in (13.93) and using (13.94)-(13.95) yields

$$\begin{aligned} MSE &= E\{(X - \hat{X})^2\} = E\{(X - \hat{X}_0 + \hat{X}_0 - \hat{X})^2\} \\ &= E\{(X - \hat{X}_0)^2\} + 2E\{(X - \hat{X}_0)(\hat{X}_0 - \hat{X})\} + E\{(\hat{X}_0 - \hat{X})^2\} \\ &= MSE_0 + \Delta, \end{aligned}$$

where

$$\Delta \triangleq E\{(\hat{X}_0 - \hat{X})^2\} = 0 \quad \text{if and only if} \quad \hat{X} = \hat{X}_0.$$

Therefore, (13.95) and (13.96) do indeed minimize MSE.

13.15 *a*) Substituting (13.106)-(13.107) into (13.101) yields

$$\int_V h_0(t-u)R_Y(v-u)du = R_{XY}(t-v) \qquad \text{for all } v \in V, t \in T$$

or, equivalently, with $\tau = t - v$ and $w = t - u$,

$$\int_{-\infty}^{\infty} h_0(w)R_Y(\tau - w)dw = R_{XY}(\tau) \qquad \text{for all } \tau,$$

which is (13.109), as desired.

b) Similarly, substituting (13.106)-(13.107) into (13.102) yields

$$MSE_0(t) = R_X(0) - \int_V h_0(t-u)R_{XY}(t-u)du \qquad \text{for } t \in T$$

$$= R_X(0) - \int_{-\infty}^{\infty} h_0(\tau)R_{XY}(\tau)d\tau \qquad (\tau = t - u).$$

13.16 Replacing $X(t)$ with $X'(t)$ and $h_0(t, u)$ with $h_0'(t, u)$ in (13.99)-(13.101), and using (13.105), we find that the optimum $h_0'(t, u)$ is specified by

$$\int_{-\infty}^{\infty} h_0'(t, u)E\{Y(v)Y(u)\}du = E\{X'(t)Y(v)\} \qquad \text{for all } v, t \in (-\infty, \infty). \qquad (*)$$

Since $X'(t) = X(t) \otimes g(t)$, then by using (13.101), (13.105), and (*) we obtain

$$E\{X'(t)Y(v)\} = E\{[X(t) \otimes g(t)]Y(v)\} = \int_{-\infty}^{\infty} g(w)E\{X(t-w)Y(v)\}dw$$

$$= \int_{-\infty}^{\infty} g(w) \int_{-\infty}^{\infty} h_0(t-w, u)E\{Y(v)Y(u)\}dudw \quad \text{(using 13.101) and (13.105))}$$

$$= \int_{-\infty}^{\infty} \Big[\int_{-\infty}^{\infty} h_0(t-w, u)g(w)dw \Big] E\{Y(v)Y(u)\}du$$

$$= \int_{-\infty}^{\infty} h_0'(t, u)E\{Y(v)Y(u)\}du \qquad \text{(using (*))}.$$

Therefore,

$$h_0'(t, u) = \int_{-\infty}^{\infty} h_0(t-w, u)g(w)dw = h_0(t, u) \otimes g(t)$$

for any $g(t)$.

13.17 Since, from (13.112), we have

$$H_{X|Y}(f) = \frac{S_{XY}(f)}{S_Y(f)} \quad \text{and} \quad H_{Y|X}(f) = \frac{S_{YX}(f)}{S_X(f)},$$

then

$$H_{X|Y}(f)H_{Y|X}(f) = \frac{|S_{XY}(f)|^2}{S_X(f)S_Y(f)} = |\rho(f)|^2,$$

where the fact that $S_{XY}(f) = S_{YX}^*(f)$ has been used.

13.18 (i) Using (13.126) and the fact that $S(t)$ and $N(t)$ are uncorrelated, we obtain

$$\begin{aligned}
MSE_0 &= E\{[S(t)-\hat{S}(t)]^2\} = E\{[d(t)\otimes S(t)+a(t)\otimes N(t)]^2\} \\
&= E\{[d(t)\otimes S(t)]^2\} + E\{[a(t)\otimes N(t)]^2\} \\
&= [R_S(\tau)\otimes r_d(\tau)]_{\tau=0} + [R_N(\tau)\otimes r_a(\tau)]_{\tau=0} \quad \text{(using (9.45))} \\
&= \int_{-\infty}^{\infty} R_S(t)r_d(t)dt + \int_{-\infty}^{\infty} R_N(t)r_a(t)dt \\
&= \int_{-\infty}^{\infty} S_S(f)|D(f)|^2 df + \int_{-\infty}^{\infty} S_N(f)|A(f)|^2 df \quad \text{(using Parseval's relation)} \\
&= MSE_S + MSE_N,
\end{aligned}$$

which is the desired result (13.130).

(ii) It follows from (13.127) that

$$h_0(t)\otimes g(t) = d(t) + \delta(t) \triangleq b(t) \quad \text{and} \quad h_0(t) = a(t).$$

Therefore, we can use (9.45) to obtain

$$\begin{aligned}
E\{[h_0(t)\otimes g(t)\otimes S(t)]^2\} &= E\{[b(t)\otimes S(t)]^2\} = [R_S(\tau)\otimes r_b(\tau)]_{\tau=0} \\
&= \int_{-\infty}^{\infty} R_S(t)r_b(t)dt = \int_{-\infty}^{\infty} S_S(f)|B(f)|^2 df = \int_{-\infty}^{\infty} S_S(f)|1+D(f)|^2 df
\end{aligned}$$

and

$$\begin{aligned}
E\{[h_0(t)\otimes N(t)]^2\} &= E\{[a(t)\otimes N(t)]^2\} = [R_N(\tau)\otimes r_a(\tau)]_{\tau=0} \\
&= \int_{-\infty}^{\infty} R_N(t)r_a(t)dt = \int_{-\infty}^{\infty} S_N(f)|A(f)|^2 df.
\end{aligned}$$

Substituting these results into the definition of SNR_0 yields the desired result (13.131).

13.19 *a*) The filter that minimizes the MSE

$$E\{[Y(t) - g(t) \otimes X(t)]^2\}$$

has transfer function $G(f)$ specified by the Wiener filter formula (13.112), with $X(t)$ and $Y(t)$ interchanged:

$$G(f) = \frac{S_{YX}(f)}{S_X(f)} = \frac{S_{XY}^*(f)}{S_X(f)}.$$

b) From the definition $N(t) \triangleq Y(t) - g(t) \otimes X(t)$, we obtain

$$R_N(\tau) = R_Y(\tau) - g(\tau) \otimes R_{XY}(\tau) - g(-\tau) \otimes R_{XY}(-\tau) + r_g(\tau) \otimes R_X(\tau).$$

Therefore, by Fourier transforming, we obtain

$$S_N(f) = S_Y(f) - G(f)S_{XY}(f) - G^*(f)S_{XY}^*(f) + |G(f)|^2 S_X(f).$$

Substituting $G(f)$ specified by (13.132a) yields the desired result (13.133).

c) As shown in part *a*, the transfer function specified by (13.132a) minimizes the mean squared value of the error $Y(t) - \hat{Y}(t)$.

13.20 (i) From (13.128)-(13.129), we see that $r(f) \gg 1$ implies that

$$|D(f)| \ll 1 \quad \text{and} \quad |A(f)| \simeq \frac{1}{|G(f)|}.$$

Therefore, (13.130) yields

$$MSE_S \ll \int_{-\infty}^{\infty} S_S(f)df = R_S(0)$$

and

$$MSE_N \simeq \int_{-\infty}^{\infty} [S_N(f) / |G(f)|^2]df.$$

Also, (13.131) yields

$$SNR_0 \simeq \frac{\int_{-\infty}^{\infty} S_S(f)df}{\int_{-\infty}^{\infty} [S_N(f) / |G(f)|^2]df}. \tag{*}$$

On the other hand, if $r(f) \ll 1$, then

$$|D(f)| \simeq 1 \quad \text{and} \quad |A(f)| \ll 1.$$

Therefore,

$$MSE_S \simeq \int_{-\infty}^{\infty} S_S(f)df = R_S(0)$$

and

$$MSE_N \ll \int_{-\infty}^{\infty} S_N(f)df = R_N(0).$$

Also, when $r(f) \ll 1$, then it follows from (13.128) and (13.129) that

$$D(f) \simeq r(f) - 1$$

and

$$A(f) \simeq r(f)/G(f).$$

Therefore, (13.131) yields

$$SNR_0 \simeq \frac{\int_{-\infty}^{\infty} |r(f)|^2 S_S(f)df}{\int_{-\infty}^{\infty} [|r(f)|^2 S_N(f)/|G(f)|^2]df}.$$

(ii) The input PSD SNR for model (13.119) is given by (13.123). At the output of any filter with transfer function $H(f) \neq 0$, we have

$$Y(t) \otimes h(t) = h(t) \otimes g(t) \otimes S(t) + h(t) \otimes N(t)$$

and, therefore, the output PSD SNR is

$$\frac{|H(f)|^2 |G(f)|^2 S_S(f)}{|H(f)|^2 S_N(f)} = \frac{|G(f)|^2 S_S(f)}{S_N(f)},$$

which is identical to the input PSD SNR. In contrast to this, if $G(f) \equiv G_0$ and

$$S_S(f) = \begin{cases} S_0, & |f| \leq B_S < B_N \\ 0, & otherwise \end{cases}$$

$$S_N(f) = \begin{cases} N_0, & |f| \leq B_N \\ 0, & otherwise, \end{cases}$$

then it follows directly from (13.268) and (13.131) that

$$SNR_i = \frac{B_S}{B_N} \frac{|G_0|^2 S_0}{N_0}$$

$$SNR_0 = \frac{|G_0|^2 S_0}{N_0} > SNR_i.$$

13.21 The PSD for $Y(t)$ is given by

$$S_Y(f) = S_S(f) + S_N(f) = \frac{a\alpha}{\alpha^2 + [2\pi(f+f_0)]^2} + \frac{a\alpha}{\alpha^2 + [2\pi(f-f_0)]^2} + N_0$$

and the cross PSD for $S(t)$ and $Y(t)$ is given by

$$S_{SY}(f) = S_S(f) = \frac{a\alpha}{\alpha^2 + [2\pi(f+f_0)]^2} + \frac{a\alpha}{\alpha^2 + [2\pi(f-f_0)]^2}.$$

Since, from the hint, we have

$$S_S(f) \simeq \frac{a\alpha}{\alpha^2 + [2\pi(f-f_0)]^2}, \quad f \geq 0$$

and $S_S(f) = S_S(-f)$, then (13.112) yields the optimum transfer function

$$H_0(f) = \frac{S_{SY}(f)}{S_Y(f)} = \frac{S_S(f)}{S_S(f) + S_N(f)} = \frac{a\alpha(\alpha^2 + [(2\pi(f-f_0)]^2)^{-1}}{a\alpha(\alpha^2 + [2\pi(f-f_0)]^2)^{-1} + N_0}$$

$$= \frac{a\alpha / N_0}{a\alpha / N_0 + \alpha^2 + [2\pi(f-f_0)]^2}, \quad f \geq 0,$$

and $H_0(f) = H_0(-f)$ (compare with (13.147)-(13.149)). The MSE_0 in (13.114) can be approximated as follows:

$$MSE_0 = \int_{-\infty}^{\infty} [S_S(f) - H_0(f) S_{SY}^*(f)] df$$

$$= \int_{-\infty}^{\infty} S_S(f)[1 - H_0(f)] df \quad (\text{since } S_{SY}(f) = S_S(f))$$

$$= \int_{-\infty}^{\infty} \frac{S_S(f) S_N(f)}{S_S(f) + S_N(f)} df = \int_{-\infty}^{\infty} H_0(f) S_N(f) df$$

$$\simeq \int_{-\infty}^{\infty} [\frac{a\alpha}{a\alpha / N_0 + \alpha^2 + [2\pi(f-f_0)]^2} + \frac{a\alpha}{a\alpha / N_0 + \alpha^2 + [2\pi(f+f_0)]^2}] df$$

$$= \frac{a\alpha}{\sqrt{a\alpha / N_0 + \alpha^2}} \exp\{|\tau| \sqrt{a\alpha / N_0 + \alpha^2}\} \cos(2\pi f_0 \tau) \Big|_{\tau = 0}$$

$$= \frac{a}{\sqrt{1 + a / \alpha N_0}}.$$

Note: α and $\beta = \alpha\sqrt{1 + a / \alpha N_0} \simeq \alpha[R_S(0) / MSE_0]$ can be taken as measures of the bandwidths of the signal and filter, respectively. Thus, as the normalized minimum MSE increases, the bandwidth of the filter decreases, approaching that of the signal.

13.22 Since

$$S_{SY}(f) = S_S(f) \quad \text{and} \quad S_Y(f) = S_S(f) + S_N(f),$$

then (13.112) reduces to

$$H_0(f) = \frac{S_{SY}(f)}{S_Y(f)} = \frac{S_S(f)}{S_S(f) + S_N(f)}$$

and (13.114), therefore, reduces to

$$MSE_0 = \int_{-\infty}^{\infty} [S_S(f) - H_0(f) S_{SY}^*(f)] df = \int_{-\infty}^{\infty} S_S(f)[1 - H_0(f)] df$$

$$= \int_{-\infty}^{\infty} \frac{S_S(f) S_N(f)}{S_S(f) + S_N(f)} df = \int_{-\infty}^{\infty} N_0 H_0(f) df = N_0 h_0(0),$$

which is the desired result (13.278). (This is the method that is used in exercise 13.21.)

13.23 With $t_0 = 0$ in example 2, we have

$$H_0(f) = \frac{b}{(2\pi f)^2 + \beta^2}$$

or

$$\overline{H}_0(s) = \frac{b}{\beta^2 - s^2} = \frac{b}{(\beta + s)(\beta - s)} = \frac{c_+}{\beta + s} + \frac{c_-}{\beta - s},$$

where

$$c_\pm = (\beta \pm s) \overline{H}_0(s) \Big|_{s = \mp\beta} = \frac{b}{2\beta}.$$

It follows from (13.154) that

$$[\overline{H}_0(s)]_+ = \frac{c_+}{\beta + s}$$

and, therefore,

$$\hat{H}_0(f) = \frac{b/2\beta}{\beta + i2\pi f}.$$

13.24 *a*) Since the time-averaged squared value of a sample path of a stationary random process is a valid norm,

$$\|X\| = \langle X^2(t) \rangle^{1/2},$$

which induces the inner product

$$(X, Y) = \langle X(t) Y(t) \rangle,$$

then we can employ the orthogonality condition

$$(X-\hat{X}_0, \hat{X}) = 0 \qquad \text{for all} \qquad \hat{X} \in M$$

to solve the minimization problem

$$\min <[X(t)-\hat{X}(t)]^2>,$$

where

$$\hat{X}(t) = \int_{-\infty}^{\infty} h(u)Y(t-u)du. \qquad (*)$$

The orthogonality condition in this case becomes

$$0 = <[X(t)-\hat{X}_0(t)]\hat{X}(t)>$$

$$= <\Big[X(t) - \int_{-\infty}^{\infty} h_0(u)Y(t-u)du\Big] \int_{-\infty}^{\infty} h(v)Y(t-v)dv>$$

$$= \int_{-\infty}^{\infty} h(v)\Big[<X(t)Y(t-v)> - \int_{-\infty}^{\infty} h_0(u)<Y(t-u)Y(t-v)>du\Big]dv,$$

which is satisfied for all $\hat{X}(t)$ if and only if it is satisfied for all $h(\cdot)$, which is so if and only if

$$<X(t)Y(t-v)> - \int_{-\infty}^{\infty} h_0(u)<Y(t-u)Y(t-v)>du = 0 \qquad \text{for all} \qquad v$$

or, equivalently,

$$\int_{-\infty}^{\infty} h_0(u)\hat{R}_Y(v-u)du = \hat{R}_{XY}(v) \qquad \text{for all} \qquad v,$$

which can be expressed as

$$h_0(v) \otimes \hat{R}_Y(v) = \hat{R}_{XY}(v) \qquad \text{for all} \qquad v.$$

Fourier transforming this necessary and sufficient condition yields

$$H_0(f)\hat{S}_Y(f) = \hat{S}_{XY}(f) \qquad \text{for all} \qquad f,$$

which has the solution (13.280).

Note: the convolution (*) is expressed in a way that streamlines this derivation. To illustrate this point, the alternative expression for this convolution is used in part *b*. See also the note in *b*.

b) Since the time-averaged mean-squared value of an asymptotically mean stationary random process is a valid norm,

$$\|X\| = <E\{X^2(t)\}>^{1/2},$$

which induces the inner product

$$(X, Y) = <E\{X(t)Y(t)\}>,$$

then we can employ the orthogonality condition

$$(X - \hat{X}_0, \hat{X}) = 0 \qquad \text{for all} \qquad \hat{X} \in M$$

to solve the minimization problem

$$\min <E\{[X(t) - \hat{X}(t)]^2\}>,$$

where

$$\hat{X}(t) = \int_{-\infty}^{\infty} h(t-u)Y(u)du.$$

The orthogonality condition in this case becomes

$$\begin{aligned}
0 &= <E\{X(t) - \hat{X}_0(t)]\hat{X}(t)\}> \\
&= <E\{\Big[X(t) - \int_{-\infty}^{\infty} h_0(t-u)Y(u)du\Big] \int_{-\infty}^{\infty} h(t-v)Y(v)dv\}> \\
&= <\int_{-\infty}^{\infty} h(t-v)\Big[E\{X(t)Y(v)\} - \int_{-\infty}^{\infty} h_0(t-u)E\{Y(u)Y(v)\}du\Big]dv> \\
&= <\int_{-\infty}^{\infty} h(w)\Big[E\{X(t)Y(t-w)\} - \int_{-\infty}^{\infty} h_0(z)E\{Y(t-z)Y(t-w)\}dz\Big]dw> \\
&\qquad\qquad\qquad\qquad\qquad (\text{using } u = t - z \text{ and } v = t - w) \\
&= \int_{-\infty}^{\infty} h(w)\Big[<E\{X(t)Y(t-w)\}> - \int_{-\infty}^{\infty} h_0(z)<E\{Y(t-z)Y(t-w)\}>dz\Big]dw,
\end{aligned}$$

which is satisfied for all $\hat{X}(t)$ if and only if it is satisfied for all $h(\cdot)$, which is so if and only if

$$<E\{X(t)Y(t-w)\}> - \int_{-\infty}^{\infty} h_0(z)<E\{Y(t-z)Y(t-w)\}>dz = 0 \qquad \text{for all} \quad w.$$

This can be rewritten as

$$\int_{-\infty}^{\infty} h_0(z)<R_Y>(w-z)dz = <R_{XY}>(w) \qquad \text{for all} \qquad w$$

or, equivalently,

$$h_0(w) \otimes <R_Y>(w) = <R_{XY}>(w).$$

Fourier transforming this necessary and sufficient condition yields

$$H_0(f)\langle S_Y\rangle(f) = \langle S_{XY}\rangle(f) \qquad \text{for all} \qquad f,$$

which has the solution (13.282).

Note: A technicality that is not mentioned in the book is that the unique solution to the minimum-norm problem obtained by orthogonal projection is guaranteed to exist only if the inner-product space is *complete*. Actually, the term *Hilbert space* is reserved for only those infinite dimensional inner-product spaces that *are* complete. Also, all finite dimensional linear spaces are complete. *Completeness* means that the space contains all its limit points (it has no open boundaries or holes). Although the space of all finite-mean-square random variables and the space of all finite-power waveforms (e.g., sample paths of a stationary processes) are indeed complete, there does not exist any linear space (let alone complete) of all jointly stationary finite-mean-square processes, or any linear space of all jointly asymptotically mean stationary processes, or any linear space of all jointly cyclostationary finite-mean-square processes. The problem is that such *joint* vector properties apparently rule out the existence of linear spaces of *all* such vectors. Nevertheless, given two processes $X(t)$ and $Y(t)$ that are jointly stationary, for example, one can indeed construct a Hilbert space of all linear combinations of $X(t)$ and $Y(t-v)$ for all v. This is all that is needed to apply orthogonal projection as done in the solution to this exercise.

13.25 The MSE in (13.99) can be expressed as (using (2.45))

$$MSE = E\{E\{[X(t)-\hat{X}(t)]^2|\Phi\}\},$$

and the corresponding orthogonality condition that minimizes MSE can be expressed as

$$E\{E\{[X(t)-\hat{X}_0(t)]\hat{X}(t)|\Phi\}\} = 0$$

or, equivalently,

$$\int_{-\infty}^{\infty} h_0(t-u)E\{E\{Y(u)Y(v)|\Phi\}\}du = E\{E\{X(t)Y(v)|\Phi\}\},$$

which -- for all v and t -- is the corresponding design equation. Let us denote

$$R_{Y|\Phi}(\tau) \triangleq E\{Y(t+\tau)Y(t)|\Phi\}$$

$$R_{XY|\Phi}(\tau) \triangleq E\{X(t+\tau)Y(t)|\Phi\}.$$

Then the design equation becomes

$$\int_{-\infty}^{\infty} h_0(t-u)E\{R_{Y|\Phi}(u-v)\}du = E\{R_{XY|\Phi}(t-v)\},$$

or, equivalently,

$$\int_{-\infty}^{\infty} h_0(w)E\{R_{Y|\Phi}(\tau-w)\}dw = h_0(\tau)\otimes E\{R_{Y|\Phi}(\tau)\} = E\{R_{XY|\Phi}(\tau)\}$$

for all τ. Fourier transforming this equation and interchanging the order of the operations of expectation and Fourier transformation yields

$$H_0(f)E\{S_{Y|\Phi}(f)\} = E\{S_{XY|\Phi}(f)\},$$

which has the solution (13.283).

13.26 (i) Since we have the observation model

$$Y(t) = S(t) + N(t),$$

where

$$S(t) = W(t) - W(t-T) \quad \text{and} \quad N(t) = \text{WGN}$$

are statistically independent, then we have the correlations

$$R_{SY}(\tau) = R_S(\tau) \quad \text{and} \quad R_Y(\tau) = R_S(\tau) + N_0\delta(\tau),$$

where (using (6.13))

$$\begin{aligned} R_S(\tau) &= E\{[W(t+\tau) - W(t+\tau-T)][W(t) - W(t-T)]\} \\ &= \alpha^2[\min\{t+\tau, t\} - \min\{t+\tau, t-T\} - \min\{t+\tau-T, t\} + \min\{t+\tau-T, t-T\}] \\ &= \begin{cases} \alpha^2(T - |\tau|), & |\tau| \le T \\ 0, & \textit{otherwise} \end{cases} \end{aligned}$$

(cf. exercise 10.4). The corresponding spectral densities are given by

$$S_{SY}(f) = \int_{-\infty}^{\infty} R_{SY}(\tau)e^{-i2\pi f\tau}d\tau = \int_{-\infty}^{\infty} R_S(\tau)e^{-i2\pi f\tau}d\tau = \alpha^2\left[\frac{\sin(\pi fT)}{\pi f}\right]^2$$

$$S_Y(f) = \alpha^2\left[\frac{\sin(\pi fT)}{\pi f}\right]^2 + N_0.$$

It follows from (13.112) that the noncausal Wiener filter is given by

$$H_0(f) = \frac{S_{SY}(f)}{S_Y(f)} = \frac{\alpha^2\left[\frac{\sin(\pi fT)}{\pi f}\right]^2}{\alpha^2\left[\frac{\sin(\pi fT)}{\pi f}\right]^2 + N_0}.$$

(ii) The optimum filter (noncausal Wiener filter) for estimating $X(t) = S(t-t_0)$ is similarly given by (13.112)

$$H_0'(f) = \frac{S_{XY}(f)}{S_Y(f)},$$

where

$$S_{XY}(f) = S_{SY}(f)e^{-i2\pi ft_0} = S_S(f)e^{-i2\pi ft_0}$$

$$S_Y(f) = S_S(f) + N_0.$$

Since $S_S(f) \ll N_0$, then we have the approximation

$$H_0'(f) = \frac{S_S(f)e^{-i2\pi f t_0}}{S_S(f) + N_0} \simeq \frac{1}{N_0} S_S(f)e^{-i2\pi f t_0},$$

and its corresponding impulse-response is

$$h_0'(t) \simeq \frac{1}{N_0} R_S(t - t_0) = \frac{\alpha^2}{N_0}[T - |t - t_0|], \qquad |t - t_0| \le T.$$

Therefore, for $t_0 > T$, this optimum filter is approximated by a causal filter.

13.27 Let the observation be $Y(t) \triangleq X(t)$; we want to estimate $Z(t) \triangleq X(t+\alpha)$ using

$$\hat{Z}(t) = \hat{X}_0(t+\alpha) = \int_{-\infty}^{t} h_0(t-u)Y(u)du$$

with the impulse response $h_0(\cdot)$ of a causal filter that minimizes the MSE. From Section 13.5 (with $X(t)$ there replaced by $Z(t)$), we obtain the following design equation for $h_0(\tau)$:

$$\int_0^\infty h_0(w)R_Y(\tau - w)dw = R_{ZY}(\tau) \qquad \text{for all} \qquad \tau \ge 0,$$

where

$$R_Y(\tau) = R_X(\tau) = ae^{-\beta|\tau|}$$

and

$$R_{ZY}(\tau) = E\{X(t+\tau+\alpha)Y(t)\} = R_X(\tau+\alpha) = ae^{-\beta|\tau+\alpha|}, \quad \alpha \ge 0.$$

Since $\tau \ge 0$ and $\alpha \ge 0$, then $R_X(\tau+\alpha) = ae^{-\beta\alpha}e^{-\beta\tau}$ and the design equation becomes

$$\int_0^\infty h_0(w)e^{-\beta|\tau - w|}dw = e^{-\beta\alpha}e^{-\beta\tau} \qquad \text{for all} \qquad \tau \ge 0.$$

Furthermore, since $|\tau| = \tau$ for $\tau \ge 0$, it is easily seen that $h_0(w) = e^{-\beta\alpha}\delta(w)$ satisfies this design equation. Thus,

$$H(f) = e^{-\beta\alpha}.$$

As an alternative approach, we can use the fact that the design equation can be expressed as

$$\int_0^\infty h_0'(w)R_Y(\tau - w)dw = R_{XY}(\tau) \qquad \text{for all} \qquad \tau \ge 0,$$

where

$$h_0'(w) = e^{\beta\alpha} h_0(w)$$

$$R_Y(\tau) = R_{XY}(\tau) = R_X(\tau).$$

The solution to this Wiener-Hopf equation is given by (13.196) with $H_0(f)$ replaced by $H_0'(f)$ and

$$\bar{S}_Y(i2\pi f) = \bar{S}_X(i2\pi f)$$

$$S_{XY}(f) = S_X(f).$$

Thus,

$$H_0'(f) = \frac{1}{[\bar{S}_X(i2\pi f)]^+}\left[\frac{S_X(f)}{[\bar{S}_X(i2\pi f)]^-}\right]_+ = \frac{1}{[\bar{S}_X(i2\pi f)]^+}\left[[\bar{S}_X(i2\pi f)]^+\right]_+$$

$$= \frac{[\bar{S}_X(i2\pi f)]^+}{[\bar{S}_X(i2\pi f)]^+} = 1$$

and, therefore,

$$H_0(f) = e^{-\alpha\beta}.$$

13.28 Since (13.198) can be expressed as

$$S_Y(f) = \bar{S}_Y(i2\pi f) = [\bar{S}_Y(i2\pi f)]^+[\bar{S}_Y(i2\pi f)]^-,$$

where

$$[\bar{S}_Y(i2\pi f)]^+ = [\bar{S}_Y(-i2\pi f)]^- = N_0^{1/2}\,\frac{\gamma + i2\pi f}{\alpha + i2\pi f},$$

then from (13.185) we have

$$G(f) = \frac{1}{[\bar{S}_Y(i2\pi f)]^+} = N_0^{-1/2}\,\frac{\alpha + i2\pi f}{\gamma + i2\pi f},$$

which verifies (13.199). Substituting (13.197) into (13.198) yields

$$S_Y(f) = \frac{2\alpha a}{\alpha^2 + (2\pi f)^2} + N_0 = N_0\frac{2\alpha a/N_0 + \alpha^2 + (2\pi f)^2}{\alpha^2 + (2\pi f)^2}.$$

Therefore, $\gamma^2 = 2\alpha a/N_0 + \alpha^2$.

From (13.200), we obtain

$$d_+ = \left[\frac{2\alpha a N_0^{-1/2}}{\gamma - i2\pi f} - \frac{d_-(\alpha + i2\pi f)}{\gamma - i2\pi f}\right]_{i2\pi f = -\alpha} = \frac{2\alpha a N_0^{-1/2}}{\gamma + \alpha}$$

$$= \frac{2\alpha a N_0^{-1/2}}{\alpha + \sqrt{2\alpha a/N_0 + \alpha^2}}.$$

Inverse Fourier transforming (13.202) yields

$$h_0(t) = N_0^{-1/2} d_+ e^{-\gamma t} u(t),$$

which implies that c in (13.203) is given by

$$c = N_0^{-1/2} d_+ = \frac{1}{N_0} \frac{2\alpha a}{\alpha + \sqrt{2\alpha a / N_0 + \alpha^2}} = \frac{1}{N_0} \frac{2a}{1 + \sqrt{2a / \alpha N_0 + 1}}.$$

The noncausal filter for $t_0 = 0$ is given by (13.150)

$$h_0'(t) = k e^{-\beta |t|},$$

where

$$k = \frac{1}{N_0} \frac{a}{\sqrt{2a / \alpha N_0 + 1}} \quad \text{and} \quad \beta^2 = 2a\alpha / N_0 + \alpha^2.$$

Hence, $\gamma = \beta$ and

$$c = \frac{2k}{1 + kN_0 / a}.$$

Thus, the optimum causal and noncausal filters have the same bandwidth but the gains are different. For high SNR ($a \gg N_0 \alpha$), $c \simeq 2k$; that is, the causal filter has twice the gain of the noncausal filter.

13.29 The performance of the noncausal filter is given in exercise 13.22 by (using (13.150))

$$MSE_0(\text{noncausal}) = N_0 h_0'(0) = \frac{a}{\sqrt{2a / \alpha N_0 + 1}},$$

and the performance of the causal filter is given by (13.204) (using the result of exercise 13.28 for c in (13.203))

$$MSE_0(\text{causal}) = N_0 h_0(0^+) = \frac{2a}{1 + \sqrt{2a / \alpha N_0 + 1}}.$$

Thus,

$$\frac{MSE_0(\text{noncausal})}{R_S(0)} = \frac{1}{\sqrt{1 + 2\lambda}}$$

$$\frac{MSE_0(\text{causal})}{R_S(0)} = \frac{2}{1 + \sqrt{1 + 2\lambda}},$$

where $\lambda = a / \alpha N_0$. For high SNR ($\lambda \gg 1$), the noncausal MSE_0 is only half the causal MSE_0, but as the SNR decreases, the noncausal MSE_0 approaches the causal MSE_0.

13.30 Since $Y(t) = X(t) + N(t)$ where $X(t)$ and $N(t)$ are statistically independent and $N(t)$ is WGN, then $R_{XY}(\tau) = R_X(\tau)$ and $R_Y(\tau) = R_X(\tau) + N_0\delta(\tau)$. It follows that the design equation (13.168) becomes

$$\int_0^\infty h_0(w)R_X(\tau - w)dw + N_0 h_0(\tau) = R_X(\tau), \quad \tau \geq 0 \tag{*}$$

and the performance formula (13.169) becomes

$$MSE_0 = R_X(0) - \int_0^\infty h_0(w)R_X(w)dw. \tag{**}$$

Substituting (*) evaluated at $\tau = 0^+$ into (**) yields the desired result (13.204).

Note: $\tau = 0^+$ is used instead of $\tau = 0$, since $\tau = 0^-$ yields $h_0(\tau) = 0$ due to causality.

13.31 Let

$$G(f) = \frac{1}{[\bar{S}_Y(i2\pi f)]^+}$$

and

$$T(f) = G^{-1}(f)e^{-i2\pi f t_0}$$

for any $t_0 > 0$. Then the impulse-response corresponding to $T(f)$ is just a delayed version of the causal stable impulse-response corresponding to $G^{-1}(f)$ and is, therefore, causal and stable. Moreover, we have (see Figure 13.4)

$$S_{\tilde{Y}}(f) = \frac{1}{|G(f)|^2}S_Z(f) = |G^{-1}(f)|^2 S_Z(f) = S_Y(f).$$

Nevertheless, if we require $\tilde{Y}(t) = Y(t)$, then we have $\tilde{Z}(t) = Z(t + t_0)$. Of course, by using a nonlinear phase (versus f) in place of $2\pi f t_0$ in $T(f)$, we can make $\tilde{Z}(f)$ differ from $Z(t)$ in a more substantial way.

13.32 Since $Y(t) = S(t) + N(t)$, where $S(t)$ and $N(t)$ are statistically independent and $N(t)$ is WGN, then $S_{SY}(f) = S_S(f) = S_Y(f) - N_0$. It follows from (13.196) that

$$\begin{aligned}
H_0(f) &= \frac{1}{[\bar{S}_Y(i2\pi f)]^+}\left[\frac{S_{SY}(f)}{[\bar{S}_Y(-i2\pi f)]^+}\right]_+ \\
&= \frac{1}{[\bar{S}_Y(i2\pi f)]^+}\left[\frac{S_Y(f)}{[\bar{S}_Y(-i2\pi f)]^+} - \frac{N_0}{[\bar{S}_Y(-i2\pi f)]^+}\right]_+ \\
&= \frac{1}{[\bar{S}_Y(i2\pi f)]^+}\left[[\bar{S}_Y(i2\pi f)]^+ - \frac{N_0}{[\bar{S}_Y(i2\pi f)]^-}\right]_+ \quad (\text{since } A/A^- = A^+)
\end{aligned}$$

$$= 1 - \frac{1}{[\bar{S}_Y(i2\pi f)]^+}\left[\frac{N_0}{[\bar{S}_Y(i2\pi f)]^-}\right]_+ \qquad (\text{since } [A^+ + B]_+ = A^+ + B_+)$$

$$= 1 - \frac{N_0^{1/2}}{[\bar{S}_Y(i2\pi f)]^+}\left[\left[\frac{N_0}{\bar{S}_Y(i2\pi f)}\right]^-\right]_+$$

$$= 1 - \frac{N_0^{1/2}}{[N_0 + \bar{S}_S(i2\pi f)]^+}\left[\left[\frac{N_0}{N_0 + \bar{S}_S(i2\pi f)}\right]^-\right]_+$$

$$= 1 - \frac{N_0^{1/2}}{[N_0 + \bar{S}_S(i2\pi f)]^+} \qquad (\text{since } [1/[1+A]^-]_+ = 1).$$

13.33 Since

$$S_{SY}(f) = S_S(f), \quad S_{NY}(f) = S_N(f) \quad \text{and} \quad S_Y(f) = S_S(f) + S_N(f),$$

then from (13.196) we obtain

$$H_1(f) = \frac{1}{[\bar{S}_Y(i2\pi f)]^+}\left[\frac{S_S(f)}{[\bar{S}_Y(-i2\pi f)]^+}\right]_+$$

$$H_2(f) = \frac{1}{[\bar{S}_Y(i2\pi f)]^+}\left[\frac{S_N(f)}{[\bar{S}_Y(-i2\pi f)]^+}\right]_+$$

and

$$H_1(f) + H_2(f) = \frac{1}{[\bar{S}_Y(i2\pi f)]^+}\left[\frac{S_S(f) + S_N(f)}{[\bar{S}_Y(-i2\pi f)]^+}\right]_+$$

$$= \frac{1}{[\bar{S}_Y(i2\pi f)]^+}[[\bar{S}_Y(i2\pi f)]^+]_+ \equiv 1.$$

13.34 The linear predictor $\hat{N}(i\,|i-1)$ of $N(i) = Y(i) - X(i)$, based on the observations $\{Y(j)\}_0^{i-1}$, is simply the difference $\hat{Y}(i\,|i-1) - \hat{X}(i\,|i-1)$ of the linear predictors of $Y(i)$ and $X(i)$, each based on the same observations $\{Y(j)\}_0^{i-1}$ (cf. exercise 13.37). But since $N(i)$ is statistically independent of $\{Y(j)\}_0^{i-1}$, then this predictor is zero: $\hat{N}(i\,|i-1) \equiv 0$. Thus, $\hat{Y}(i\,|i-1) \equiv \hat{X}(i\,|i-1)$.

As an alternative approach, we observe that, for the model (13.207) in which $N(j)$ is WGN, we have

$$E\{Y(i)Y(i-j)\} \equiv E\{X(i)Y(i-j)\}, \quad j = 1, 2, \ldots, i$$

and, therefore, the difference of (13.210) and (13.211) satisfies

$$E\{[\hat{Y}(i\,|i-1) - \hat{X}(i\,|i-1)]Y(i-j)\} = 0, \quad j=1, 2, \ldots, i.$$

But since both $\hat{Y}(i\,|i-1)$ and $\hat{X}(i\,|i-1)$ are linear combinations of the linearly independent variables $\{Y(i-j)\}_{j=1}^{i}$, then the difference $\hat{Y}(i\,|i-1) - \hat{X}(i\,|i-1)$ must be identically zero; otherwise the linear independence property would be violated.

13.35 Using (13.210), (13.212), and (13.214), we obtain for $j < i$

$$\begin{aligned} R_Z(i, j) &= E\{Z(i)Z(j)\} = E\{[Y(i) - \hat{Y}(i\,|i-1)][Y(j) - \hat{Y}(j\,|j-1)]\} \\ &= E\{[Y(i) - \hat{Y}(i\,|i-1)]Y(j)\} - \sum_{k=0}^{j-1} f(j, k)E\{[Y(i) - \hat{Y}(i\,|i-1)]Y(k)\} \\ &= 0 - 0 = 0. \end{aligned}$$

Since, $R_Z(i, j) = R_Z(j, i)$, then we also have $R_Z(i, j) = 0$ for $j > i$. Thus,

$$R_Z(i, j) = \begin{cases} R_Z(i, i), & j = i \\ 0, & j \neq i, \end{cases}$$

which is the desired result (13.213).

13.36 From (13.220), we have

$$k(i)R_Z(i, i) = R_{XZ}(i, i),$$

or, equivalently,

$$E\{[X(i) - k(i)Z(i)]Z(i)\} = 0,$$

which implies that the projection $k(i)Z(i)$ of $X(i)$ onto $Z(i)$ is an orthogonal projection.

13.37 Let Y be a subspace of the space containing the vectors W_1 and W_2, and let $\hat{W}_1$ and $\hat{W}_2$ be the orthogonal projections of W_1 and W_2 onto Y. Then

$$\begin{aligned} (W_1 - \hat{W}_1, W) &= 0 \qquad W \in Y \\ (W_2 - \hat{W}_2, W) &= 0 \qquad W \in Y. \end{aligned}$$

It follows that

$$\begin{aligned} (c_1W_1 + c_2W_2 - [c_1\hat{W}_1 + c_2\hat{W}_2], W) &= (c_1[W_1 - \hat{W}_1] + c_2[W_2 - \hat{W}_2], W) \\ &= c_1(W_1 - \hat{W}_1, W) + c_2(W_2 - \hat{W}_2, W) = 0, \qquad W \in Y. \end{aligned}$$

Therefore, $c_1\hat{W}_1 + c_2\hat{W}_2$ is the orthogonal projection of $c_1W_1 + c_2W_2$ onto Y.

13.38 From the orthogonality condition,

$$E\{[\mathbf{W}(i+1)-\hat{\mathbf{W}}(i+1|i)]Z(k)\} = 0, \quad k = 0, 1, \ldots, i$$

for the predictor $\hat{\mathbf{W}}(i+1|i)$ based on the innovations $\{Z(k)\}$, we have

$$E\{\mathbf{W}(i+1)Z(k)\} = \mathbf{R}_{\mathbf{W}Z}(i+1, k) = E\{\hat{\mathbf{W}}(i+1|i)Z(k)\}$$

$$= \sum_{j=0}^{i} \mathbf{h}_*(i, j)E\{Z(j)Z(k)\} = \sum_{j=0}^{i} \mathbf{h}_*(i, j)R_Z(k, j), \quad k = 0, 1, \ldots, i,$$

which is the desired design equation (13.228). Substituting (13.213) into (13.228) yields

$$\sum_{j=1}^{i} R_Z(k, k)\delta_{k-j}\mathbf{h}_*(i, j) = R_Z(k, k)\mathbf{h}_*(i, k) = \mathbf{R}_{\mathbf{W}Z}(i+1, k), \quad k = 0, 1, \ldots, i,$$

which has the solution

$$\mathbf{h}_*(i, k) = R_Z^{-1}(k, k)\mathbf{R}_{\mathbf{W}Z}(i+1, k).$$

Therefore, (13.227) yields the state prediction

$$\hat{\mathbf{W}}(i+1|i) = \sum_{j=0}^{i} \mathbf{h}_*(i, j)Z(j) = \sum_{j=0}^{i} R_Z^{-1}(j, j)\mathbf{R}_{\mathbf{W}Z}(i+1, j)Z(j),$$

which is the desired result (13.229).

13.39 From (13.223), we obtain

$$\mathbf{R}_{\mathbf{W}Z}(i+1, j) = E\{\mathbf{W}(i+1)Z(j)\} = \mathbf{A}(i)E\{\mathbf{W}(i)Z(j)\} + \mathbf{b}(i)E\{V(i)Z(j)\}$$

$$= \mathbf{A}(i)\mathbf{R}_{\mathbf{W}Z}(i, j), \quad j \le i,$$

which is the desired equation (13.276). Substituting (13.276) into (13.229) yields the recursion (13.230):

$$\hat{\mathbf{W}}(i+1|i) = \sum_{j=0}^{i-1} R_Z^{-1}(j, j)\mathbf{R}_{\mathbf{W}Z}(i+1, j)Z(j) + R_Z^{-1}(i, i)\mathbf{R}_{\mathbf{W}Z}(i+1, i)Z(i)$$

$$= \mathbf{A}(i)\sum_{j=0}^{i-1} R_Z^{-1}(j, j)\mathbf{R}_{\mathbf{W}Z}(i, j)Z(j) + \mathbf{k}(i+1, i)Z(i)$$

$$= \mathbf{A}(i)\hat{\mathbf{W}}(i|i-1) + \mathbf{k}(i+1, i)Z(i),$$

where $\mathbf{k}(i+1, i)$ is defined by (13.231).

13.40 Substituting (13.276) into (13.231) yields (using definition (13.237))

$$\mathbf{k}(i+1, i) = R_Z^{-1}(i, i)\mathbf{A}(i)\mathbf{R}_{\mathbf{W}Z}(i, i) = \mathbf{A}(i)\mathbf{k}(i),$$

which is (13.234) as desired.

It follows from (13.276), (13.232)-(13.233), and (13.227) that

$$\begin{aligned}
\mathbf{R}_{\mathbf{W}Z}(i+1, i) &= \mathbf{A}(i)\mathbf{R}_{\mathbf{W}Z}(i, i) = \mathbf{A}(i)E\{[\tilde{\mathbf{W}}(i|i-1) + \hat{\mathbf{W}}(i|i-1)]Z(i)\} \\
&= \mathbf{A}(i)E\{\tilde{\mathbf{W}}(i|i-1)Z(i)\} + \mathbf{A}(i)\sum_{j=0}^{i-1} \mathbf{h}_*(i-1, j)E\{Z(j)Z(i)\} \\
&= \mathbf{A}(i)E\{\tilde{\mathbf{W}}(i|i-1)[\mathbf{c}(i)\tilde{\mathbf{W}}(i|i-1) + N(i)]^T\} + 0 \\
&= \mathbf{A}(i)[\mathbf{R}_{\tilde{\mathbf{W}}}(i|i-1)\mathbf{c}^T(i) + E\{\tilde{\mathbf{W}}(i|i-1)N(i)\}] \\
&= \mathbf{A}(i)\mathbf{R}_{\tilde{\mathbf{W}}}(i|i-1)\mathbf{c}^T(i) + E\{\mathbf{W}(i)N(i)\} - E\{\hat{\mathbf{W}}(i|i-1)N(i)\} \\
&= \mathbf{A}(i)\mathbf{R}_{\tilde{\mathbf{W}}}(i|i-1)\mathbf{c}^T(i) + 0 - 0,
\end{aligned}$$

and it follows from (13.232b) that

$$\begin{aligned}
R_Z(i, i) &= E\{[\mathbf{c}(i)\tilde{\mathbf{W}}(i|i-1) + N(i)][\mathbf{c}(i)\tilde{\mathbf{W}}(i|i-1) + N(i)]^T\} \\
&= \mathbf{c}(i)\mathbf{R}_{\tilde{\mathbf{W}}}(i|i-1)\mathbf{c}^T(i) + N_0.
\end{aligned}$$

Therefore, (13.231) can be expressed as

$$\mathbf{k}(i+1|i) = \mathbf{A}(i)R_Z^{-1}(i, i)\mathbf{R}_{\mathbf{W}Z}(i, i) = \frac{\mathbf{A}(i)\mathbf{R}_{\tilde{\mathbf{W}}}(i|i-1)\mathbf{c}^T(i)}{\mathbf{c}(i)\mathbf{R}_{\tilde{\mathbf{W}}}(i|i-1)\mathbf{c}^T(i) + N_0},$$

which, together with (13.234), yields the desired result (13.235a).

13.41 From (13.236), we obtain

$$\hat{\mathbf{W}}(i|i-1) = \hat{\mathbf{W}}(i|i) - \mathbf{k}(i)Z(i)$$

which, upon substitution into (13.230), yields

$$\begin{aligned}
\hat{\mathbf{W}}(i+1|i) &= \mathbf{A}(i)\hat{\mathbf{W}}(i|i) - \mathbf{A}(i)\mathbf{k}(i)Z(i) + \mathbf{k}(i+1, i)Z(i) \\
&= \mathbf{A}(i)\hat{\mathbf{W}}(i|i) \qquad \text{(using (13.234))},
\end{aligned}$$

which is the desired relation (13.238).

13.42 Substituting (13.232b) into (13.236) and subtracting $\mathbf{W}(i)$ from both sides yields

$$\hat{\mathbf{W}}(i|i) = \hat{\mathbf{W}}(i|i-1) + \mathbf{k}(i)\mathbf{c}(i)\tilde{\mathbf{W}}(i|i-1) + N(i)\mathbf{k}(i).$$

Therefore, definition (13.277) can be expressed as

$$\begin{aligned}
\tilde{\mathbf{W}}(i|i) \triangleq \mathbf{W}(i) - \hat{\mathbf{W}}(i|i) &= \mathbf{W}(i) - \hat{\mathbf{W}}(i|i-1) - \mathbf{k}(i)\mathbf{c}(i)\tilde{\mathbf{W}}(i|i-1) - N(i)\mathbf{k}(i) \\
&= [\mathbf{I} - \mathbf{k}(i)\mathbf{c}(i)]\tilde{\mathbf{W}}(i|i-1) - N(i)\mathbf{k}(i),
\end{aligned}$$

which is (13.286), as desired. Using this equation, the correlation matrix of $\tilde{\mathbf{W}}(i\,|i)$ can be expressed as

$$\mathbf{R}_{\tilde{\mathbf{W}}}(i\,|i) = E\{\tilde{\mathbf{W}}(i\,|i)\tilde{\mathbf{W}}^T(i\,|i)\}$$

$$= [\mathbf{I} - \mathbf{k}(i)\mathbf{c}(i)]E\{\tilde{\mathbf{W}}(i\,|i-1)\tilde{\mathbf{W}}^T(i\,|i-1)\}[\mathbf{I} - \mathbf{k}(i)\mathbf{c}(i)]^T + \mathbf{k}(i)\mathbf{k}^T(i)E\{N^2(i)\}$$

$$(\text{since } E\{\tilde{\mathbf{W}}(i\,|i-1)N(i)\} = 0)$$

$$= [\mathbf{I} - \mathbf{k}(i)\mathbf{c}(i)]\mathbf{R}_{\tilde{\mathbf{W}}}(i\,|i-1)[\mathbf{I} - \mathbf{k}(i)\mathbf{c}(i)]^T + N_0\mathbf{k}(i)\mathbf{k}^T(i),$$

which is (13.279), as desired. Substituting (13.235a) into (13.279) yields

$$\mathbf{R}_{\tilde{\mathbf{W}}}(i\,|i) = [\mathbf{I} - \frac{\mathbf{R}_{\tilde{\mathbf{W}}}(i\,|i-1)\mathbf{c}^T(i)\mathbf{c}(i)}{\mathbf{c}(i)\mathbf{R}_{\tilde{\mathbf{W}}}(i\,|i-1)\mathbf{c}^T(i) + N_0}]\mathbf{R}_{\tilde{\mathbf{W}}}(i\,|i-1)$$

$$\times [\mathbf{I} - \frac{\mathbf{R}_{\tilde{\mathbf{W}}}(i\,|i-1)\mathbf{c}^T(i)\mathbf{c}(i)}{\mathbf{c}(i)\mathbf{R}_{\tilde{\mathbf{W}}}(i\,|i-1)\mathbf{c}^T(i) + N_0}]^T + N_0\frac{\mathbf{R}_{\tilde{\mathbf{W}}}(i\,|i-1)\mathbf{c}^T(i)\mathbf{c}(i)\mathbf{R}_{\tilde{\mathbf{W}}}^T(i\,|i-1)}{[\mathbf{c}(i)\mathbf{R}_{\tilde{\mathbf{W}}}(i\,|i-1)\mathbf{c}^T(i) + N_0]^2}$$

$$= \mathbf{R}_{\tilde{\mathbf{W}}}(i\,|i-1) - \frac{\mathbf{R}_{\tilde{\mathbf{W}}}(i\,|i-1)\mathbf{c}^T(i)\mathbf{c}(i)\mathbf{R}_{\tilde{\mathbf{W}}}(i\,|i-1)}{N_0 + \mathbf{c}(i)\mathbf{R}_{\tilde{\mathbf{W}}}(i\,|i-1)\mathbf{c}^T(i)},$$

which is (13.280), as desired. Substituting (13.238) into (13.233) and then substituting (13.223) into the result yields

$$\tilde{\mathbf{W}}(i\,|i-1) = \mathbf{W}(i) - \mathbf{A}(i-1)\hat{\mathbf{W}}(i-1\,|i-1)$$

$$= \mathbf{A}(i-1)\mathbf{W}(i-1) + \mathbf{b}(i-1)V(i-1) - \mathbf{A}(i-1)\hat{\mathbf{W}}(i-1\,|i-1)$$

$$= \mathbf{A}(i-1)\tilde{\mathbf{W}}(i-1\,|i-1) + \mathbf{b}(i-1)V(i-1),$$

from which we obtain the correlation matrix of $\tilde{\mathbf{W}}(i\,|i-1)$:

$$\mathbf{R}_{\tilde{\mathbf{W}}}(i\,|i-1) = E\{\tilde{\mathbf{W}}(i\,|i-1)\tilde{\mathbf{W}}^T(i\,|i-1)\}$$

$$= \mathbf{A}(i-1)E\{\tilde{\mathbf{W}}(i-1\,|i-1)\tilde{\mathbf{W}}^T(i-1\,|i-1)\}\mathbf{A}^T(i-1) + \mathbf{b}(i-1)\mathbf{b}^T(i-1)E\{V^2(i-1)\}$$

$$(\text{since } E\{\tilde{\mathbf{W}}(i-1\,|i-1)V(i-1)\} = 0)$$

$$= \mathbf{A}(i-1)\mathbf{R}_{\tilde{\mathbf{W}}}(i-1\,|i-1)\mathbf{A}^T(i-1) + \mathbf{b}(i-1)\mathbf{b}^T(i-1),$$

which is the desired result (13.282). Substituting (13.235a) into (13.280) yields the alternative form of the recursion:

$$\mathbf{R}_{\tilde{\mathbf{W}}}(i\,|i) = \mathbf{R}_{\tilde{\mathbf{W}}}(i\,|i-1) - \mathbf{k}(i)\mathbf{c}(i)\mathbf{R}_{\tilde{\mathbf{W}}}(i\,|i-1) = [\mathbf{I} - \mathbf{k}(i)\mathbf{c}(i)]\mathbf{R}_{\tilde{\mathbf{W}}}(i\,|i-1).$$

13.43 *a*) From (13.248), we obtain

$$R_{\tilde{W}}(i\,|i-1) = \frac{N_0 k(i)}{c - c^2 k(i)}.$$

Substituting $R_{\tilde{W}}(i\,|i-1)$ and (13.248) into (13.250) yields

$$\frac{N_0 k(i)}{c - c^2 k(i)} = \frac{a^2 N_0}{c} k(i-1) + b^2, \quad i \geq 2$$

or, equivalently,

$$k(i) = \frac{a^2 k(i-1) + b^2 c/N_0}{1 + (bc)^2/N_0 + a^2 ck(i-1)}, \quad i \geq 2,$$

which is (13.252), as desired.

b) Substituting the initial conditions $R_W(0) = 0$ and $\tilde{W}(0|0) = 0$ into (13.251) yields

$$R_{\tilde{W}}(1\,|0) = a^2[R_W(0) - \tilde{W}^2(0|0)] + b^2 = b^2.$$

Substituting this result into (13.248) yields

$$k(1) = \frac{cR_{\tilde{W}}(1\,|0)}{N_0 + c^2 R_{\tilde{W}}(1\,|0)} = \frac{cb^2}{N_0 + (cb)^2},$$

which is the desired result (13.253).

c) Substituting (13.236) into (13.238) with i replaced by $i-1$ yields

$$\hat{W}(i\,|i-1) = a\hat{W}(i-1\,|i-1) = a\hat{W}(i-1\,|i-2) + ak(i-1)Z(i-1).$$

By comparing this difference equation with (13.245), we see that these two difference equations are identical except for a factor of a by which the input to the predictor is scaled. Therefore, the impulse response functions for these two systems are related by $g(i) = ah(i)$, and $h(i)$ is solved for in exercise 13.5.